MATHEU
Identification, Motivation and Support of Mathematical Talents in European Schools

MANUAL

Volume 3

Editor
Gregory Makrides, INTERCOLLEGE, Cyprus

Published by
MATH.EU Project

ISBN 9963-634-31-1

MATHEU
Identification, Motivation and Support of Mathematical Talents in European Schools

MANUAL

Volume 3 (Level 2)

Editor
Gregory Makrides, INTERCOLLEGE, Cyprus

Published by
MATH.EU Project

ISBN 9963-634-31-1

This project has been carried out with the support of the European Commission within the framework of the Socrates Programme. Information expressed reflects the views only of the MATHEU project partnership. The European Commission cannot be held responsible for any use made of this information.

Coordinator and Editor contact address

Gregory Makrides
makrides.g@ucy.ac.cy
makrides.greg@usa.net
makrides.g@intercollege.ac.cy
cms@cms.org.cy
Tel. +357-99641843

MATHEU Partners

- **INTERCOLLEGE / School of Education- Coordinating Institution, CYPRUS**
  Gregory Makrides
  Emilios Solomou
  Michalinos Zembylas
  Andreas Savva
  Elena Michael
- **Institute of Mathematics-Academy of Sciences, BULGARIA**
  Petar Kenderov
  Sava Grozdev
- **University of Cyprus, CYPRUS**
  Athanasios Gagatsis
  Costas Christou
- **Charles University, CZECH REPUBLIC**
  Jarmila Novotna
  Marie Hofmannova
  Jaroslav Zhouf
- **University of Duisburg, GERMANY**
  Werner Haussmann
- **University of Crete, GREECE**
  Michalis Lambrou
- **University of Palermo, ITALY**
  Filippo Spagnolo
- **North University of Baia Mare, ROMANIA**
  Vasile Berinde
- **University of Miskolc, HUNGARY**
  Péter Körtesi
  Jenő Szigeti

MATHEU Contributors

- **Bulgarian Union of Mathematicians**
  Svetoslav Bilchev
- **Romanian Math Society**
  Mircea Becheanu
- **European Math Society**
  Tony Gardiner , Mina Teicher
- **MASSEE**
  Emilia Velikova
- **Cyprus Mathematical Society**
  Andreas Philippou
- **Union of Mathematicians of Cyprus**
  Marios Antoniades
- **Czech Mathematical Society**
  Jaroslav Svrcek, Vaclav Sykora
- **Hellenic Math Society**
  Constantinos Salaris
- **Italian Math Society**
  Franco Favilli
- **Hungarian Mathematical Society**
  Péter Madarász

Table of Contents

***Any mathematical mistakes found in this volume are the responsibility of the author of each section or ladder. We invite the readers to inform the editor and coordinator for any mistakes found.**

DIRICHLET PRINCIPLE

Sava Grozdev
Institute of Mathematics, Bulgarian Academy of Sciences

A fundamental rule, i.e. a principle, states: **if *m* objects are distributed into *n* groups and *m* > *n*, then at least two of the objects are in one and the same group.**

People from different countries call this principle differently. For example in France it is known as "the drawers' principle", in England – as "the pigeon-hole principle", while in Bulgaria and Russia – as the Dirichlet principle. The principle is connected with the name of the great German mathematician *Gustav Lejeune-Dirichlet* (1805 – 1859) although it has been well known quite before him. The merit of Dirichlet is not in discovering the above trivial fact but in applying it to solve numerous interesting problems in Number Theory. Dirichlet himself did not settle parrots or rabbits into cages and did not distribute boxes into drawers either. In one of his scientific works he formulated a method of reasoning and based it on a principle, which took his name later. It was on the occasion of the distribution of the prime numbers in arithmetic progressions. The following theorem belongs to Dirichlet: **the sequence *an* + *b*, where *a* and *b* are relatively prime, contains infinitely many prime numbers.** This theorem is not under discussion in the present note.

Using the "language of the drawers", the Dirichlet principle establishes the existence of a drawer with certain properties. Thus, it is a statement for existence in fact. However, the principle does not propose an algorithm to find a drawer with desired properties and consequently it is of non-constructive character. Namely, constructive profs of existence stand closer to reasoning and are more convincing. It seems that this is the main reason for the unexpectedness of numerous applications of the Dirichlet principle. The following problems are dedicated to such applications after additional reasoning.

Problem 1. Let a, b, c and d be integers. Prove that the product

$$(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)$$

is divisible by 12.

Solution: The six factors of the product under consideration are formed by all the possible couples of the four integers. At least two of the four integers coincide modulo 3 by the Dirichlet principle. Their difference therefore is a multiple of 3. Now either at least three of the four integers have the same parity, in which case three of the differences are even, or the four numbers have the property that two of them are odd and two are even.

Problem 2. A student buys 17 pencils of 4 different colours. Find the greatest possible value of n to be sure that the student has bought at least n pencils of the same colour.

Solution: If the student has bought at most 4 pencils of the same colour, then the total number of the bought pencils is 4.4 = 16 at most. This is a contradiction because 16 < 17. Consequently $n > 4$ and the least possible value is $n = 5$. The drawers in this case are the different colours of pencils, i.e. they are 4. Putting 17 pencils into 4 drawers we obtain at least one drawer with not less than 5 pencils.

In the solution of the last problem it is used the following more general form of the Dirichlet principle: **if *m* subjects are distributed into *n* groups and *m* > *nk*, where *k* is a natural number, then at least *k* + 1 subjects fall into one of the groups.**

Problem 3. 145 points are taken in a rectangle with dimensions 4 m x 3 m. Is it possible to cover at least 4 points by a square with dimensions 50 cm x 50 cm?
Solution: The answer is positive. It is enough to divide the rectangle into 48 squares with dimensions 50 cm x 50 cm by lines parallel to the sides of the rectangle. The Dirichlet principle implies that at least 4 points are in one of the squares.

Problem 4. A 5 x 5 square is divided into 25 unit squares, which are coloured in blue or red. Prove that there exist 4 monochromatic unit squares which lie in the intersection of 2 rows and 2 columns of the initial square.
Solution: Firstly consider the unit squares of a column as drawers. It follows by the Dirichlet principle that one of the colours is dominating in the chosen column. Analogously, one of the colours is dominating in each of the other columns. Now the drawers are the 5 columns, while the subjects to be distributed are the dominating colours of all columns. It follows by the Dirichlet principle again that the dominating colour is one and the same in 3 columns at least. Thus, there are 3 columns and each of them is with 3 monochromatic unit squares at least. Assume that the common colour is blue. Further, enumerate the rows of the initial square by the integers from 1 to 5 and consider 5 drawers enumerating them by the same integers. Juxtapose the numbers of the corresponding rows to the blue unit squares in the 3 columns under consideration. The problem is reduced to a distribution of 9 integers (or more) among the integers 1, 2, 3, 4 and 5 into 5 drawers. Each integer should be put into the drawer with the corresponding number. We have to prove that there are 2 drawers such that each of them is with 2 integers in it at least. At first notice that a drawer exists with 2 integers at least. Two cases are possible. In the first one assume that a drawer exists with 3 integers. Then the 6 remaining integers should be distributed into in the remaining 4 drawers. It follows by the Dirichlet principle that one of them contains 2 integers at least. This drawer together with the drawer with 3 integers solves the problem. In the second case consider a drawer with 2 integers. The remaining 7 integers should be distributed into the remaining 4 drawers. One of them contains 2 integers according to the Dirichlet principle and this solves the problem again.

Problem 5. Given is a 5 x 41 rectangle. It is divided into 205 unit squares, which are coloured in blue or red. Prove that there exist 9 monochromatic unit squares which lie in the intersection of 3 rows and 3 columns of the initial square.
Solution: As in the previous problem one of the colours is dominating in all the 41 columns. At least 21 columns are with one and the same dominating colour since the colours are 2. Assume that this colour is blue. There are 3 blue unit squares at least in each of the 21 columns under consideration. Enumerate the blue unit squares by the integers from 1 to 5 respecting the rows which contain them. Thus, a triplet is juxtaposed to each of the 21 columns. Each triplet is formed by the numbers of the blue unit squares. The total number of the triplets is equal to 21. On the other hand all possible triplets are 10 using the integers 1, 2, 3, 4 and 5. It follows by the Dirichlet principle that 3 of them coincide at least. Thus, we are done.

Problem 6. 44 queens are located on an 8 x 8 chess board. Prove that each of them beats at least one of the others.
Solution: Each queen controls 21 fields at least. Considering the field on which a queen is situated we get at least 22 fields of control. Suppose that there is one queen which does not beat any other. It controls 22 fields. The remaining fields are 64 – 22 = 42. We

have 43 other queens and it follows by the Dirichlet principle that at least one of them is situated on a beat field by the queen under consideration. This is a contradiction.

Problem 7. Find the maximal number of kings on an ordinary chess board in a way that no two of them beat each other.
Solution: Each king controls 4 fields at least. The number of the controlled fields by a king is 4 exactly when the king is situated at one of the 4 vertices of the chess board (otherwise the king controls more than 4 fields). Divide the board into 16 squares 4 x 4. It follows that it is not possible to situate more than 16 kings in a way that the condition of the problem is realized, because two kings should not be in one and the same 4 x 4 square. An example of 16 is given below:

X		X		X		X	
X		X		X		X	
X		X		X		X	
X		X		X		X	

Problem 8. Prove that there exist 2 integers among 12 two-digit different positive integers such that their difference is a two-digit integer with coinciding digits.
Solution: If the drawers are the remainders modulo 11, then it follows by the Dirichlet principle that at least 2 of the integers are with the same remainder. The difference of these 2 integers is divisible modulo 11. At the same time each two-digit integer has coinciding digits when it is divisible by 11.

Problem 9. The natural numbers from 1 to 10 are written down in a column one after the other. Each of them is summed up with the number of its position in the column. Prove that at least 2 sums end with the same digit.
Solution: Assume that all sums end with different digits. Then each of the last digits is equal to exactly one of the digits 0, 1, ..., 9. Otherwise the Dirichlet principle implies that 2 of the last digits coincide. It follows by the assumption that the sum of the sums ends with the same digit as the sum $1 + 2 + ... + 9 = 45$ does, i.e. in the digit 5. On the other hand the sum of the integers from 1 to 10 is equal 55. The sum of the numbers of the positions with the column is equal to 55, too. Thus the sum of the sums is equal to 110, which ends with 0. This is a contradiction to the assumption.

Problem 10. Prove that there exist a natural number which is multiple of 2004 and its decimal representation contains 0-s and 1-s only.
Solution: Consider 2005 integers which decimal representations contain only 1, 2, … , or 2005 ones respectively, i.e. consider the integers: 1, 11, 111, ... , 11111...1. It follows by the Dirichlet principle that at least 2 of them have the same remainder modulo 2004. Their positive difference contains 0-s and 1-s only.

Problem 11. Prove that there exist 2 integers among 52 non negative integers such that their sum or difference is a multiple of 100. Is the assertion valid for 51 non negative integers?

Solution: Consider the integers from 0 to 99, which are the possible remainders modulo 100. Take 51 drawers and put the integers with remainder 0 into the first one, put the integers with remainder 1 or 99 into the second, the integers with remainder 2 or 98 – into the third, and so on, the integers with remainder 49 or 51 – into the fiftieth, the integers with remainder 50 – into the fifty first. It follows by the Dirichlet principle that at least 2 integers fall into one and the same drawer. If both integers have the same remainder then their difference is a multiple of 100. Otherwise the sum of their remainders is equal to 100 and consequently the sum of the integers themselves is a multiple of 100. The assertion is not valid for 51 non negative integers as shows the following example: 0, 1, 2, ... , 50.

Problem 12. Given are 2006 arbitrary positive integers and each of them is not divisible by 2006. Prove that the sum of several of them is divisible by 2006.
Solution: Denote the given integers by $a_1, a_2, ..., a_{2006}$ and consider the following 2006 integers:

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, ..., a_1 + a_2 + ... + a_{2006}.$$

The number of the possible remainders modulo 2006 is equal to 2006. If one of the remainders is equal to 0, then the problem is solved. Of course the remainder of the first number in the sequence is different from 0 according to the condition of the problem. If all the remainders are different from 0, then it follows by the Dirichlet principle that at least 2 of them are equal. In this case it is enough to consider the difference of these 2 numbers.

Problem 13. Given the integers 1, 2, ..., 200 and 101 of them are chosen. Prove that there exist 2 integers among the chosen ones such that the one of them divides the other.
Solution: If a is odd and less than 200, denote the set {a, 2a, 4a, 8a, 16a, 32a, 64a, 128a} by A. For each integer from 1 to 200 there exists an odd integer a < 200 such that the set A contains this integer. Since the number of the sets A is equal to 100 (this number is equal to the number of the odd integers from 1 to 200) and the number of the chosen integers is equal to 101, then it follows by the Dirichlet principle that at least 2 integers fall into one and the same set. On the other hand if 2 integers are in one and the same set, then the one of them divides the other.

Problem 14. Given are 986 different positive integers not greater than 1969. The greatest of them is odd. Prove that there exist 3 integers among the given ones such that the one is equal to the sum of the others.
Solution: Denote the greatest integer by a. It is odd according to the condition of the problem. Consider the differences between a and the other integers. Their number is equal to 985 and all of them are different. Together with the given integers there are 1971 integers totally. Since 1971 > 1969 it follows by the Dirichlet principle that at least 2 of the integers are equal. Consequently one of the differences coincides with one of the given 986 integers, i.e. $a - b = c$. We have that $b \neq c$, because a is odd. Thus $a = b + c$ and we are done. Note that the number 986 of the given integers is essential. If the number is 985, then chose all odd integers which are not greater than 1969. Their number is exactly 985. On the other hand the sum of 2 odd integers is even and consequently the assertion of the problem is not valid in this case.

Problem 15. Given a 10 x 10 chess-board and positive integers are written down on its fields in such a way that the difference of any two horizontal or vertical neighbours does not exceed 5. Prove that at least 2 of the integers on the chess board are equal.

Solution: Denote the greatest and the smallest of the written integers by *a* and *b*, respectively. If no 2 integers are equal, then it follows by the condition of the problem that $a - b \geq 99$. Connect *a* and *b* by the shortest way moving through horizontal and vertical fields only. The maximal length of the way is 18 fields (9 horizontal and 9 vertical). Thus, *a* is reached from *b* by adding 18 integers which differ by at most 5 (if neighbours), i.e. $a \leq b + 18.5$. Consequently $b + 90 \geq b + 99$. The last inequality is impossible, which implies that at least 2 of the integers are equal.

Problem 16. The integers from 1 to n^2 are written down arbitrarily on the fields of an *n* x *n* chess board. Consider the following assertion: There exist 2 fields with a common side such that the difference of the integers on them is greater than 5. Prove that

a) The assertion is not always true for $n = 5$;

b) The assertion is always true for n > 5.

Solution: a) Take a chess board 5 x 5 and write down the integers from 1 to 5 on the first row, write down the integers from 6 to 10 on the second row, the integers from 11 to 15 – on the third row, the integers from 16 to 20 – on the fourth row and write down the integers from 21 to 25 on the fifth row. The maximal difference is equal to 5 in this example.

b) Analogously to the previous problem the field with the integer n^2 could be reached from the field with the integer 1 by horizontal and vertical moves through $2(n - 1)$ fields at most. The integer 1 is increased by $n^2 - 1$ and it follows by the Dirichlet principle that there exists a step at which the increase is not less than $\frac{n^2-1}{2(n-1)} = \frac{n+1}{2}$.

When $n \geq 10$ the number $\frac{n+1}{2}$ is greater than 5 and consequently the assertion is always true in this case. When $n = 9$ the increase from 1 to 81 is equal to 81 – 1 = 80 and it is realized for 16 steps at most (8 horizontal moves and 8 vertical ones). If the number of the steps is at most 15 then the Dirichlet principle implies that there exists a step at which the increase is not less than $\frac{80}{15}$. Again the number is greater than 5. Assume that there are exactly 16 steps. Now 80 : 16 = 5 and if there is a step at which the increase is less than 5, then the total increase is 76 at least at the remaining 15 steps. It follows by the Dirichlet principle that there exists a step at which the increase is not less than 76 : 15, i.e. the increase is greater than 5. Finally consider the case when there is neither a step with an increase which is greater than 5 nor a step with an increase which is less than 5. Thus the increase is equal to 5 at each step. This means that starting from 1 the integers in the successive fields are 6, 11, 16, 21 and so on. On the other hand the way between 1 and 81 under consideration is not the only one. There are several other ways to reach 81 starting from 1. The integers on the corresponding fields of such a way should not be the same. This means that the increase is not equal to 5 at each step and one can repeat the above reasoning. It follows that the assertion is true for $n = 9$ too. By similar considerations the assertion could be proved for the cases $n = 6$, 7 and 8.

Remark. It is valid the following general fact: If the integers from 1 to n^2 $(n \geq 2)$ are written down on the fields of an *n* x *n* chess board, then there exist 2 fields with a common side such that the difference of the integers on them is not less than *n*. (Gerver,

M. L., A problem with integers in a table (in Russian), Qwant, **12**, 1971, 24 – 27). The proof of this fact is rather complicated but the main idea is involved in the proof of the following:

Problem 17. If the integers from 1 to 16 are written down on the fields of a 4 x 4 chess board, then there exist 2 fields with a common side such that the difference of the integers on them is not less than 4.

Solution:

1	**2**	*	
3	*		
*			*
		*	**4**

Consider the position of the integers 1, 2, 3, 4 and put stars on their neighbour fields as shown. No matter how the integers 1, 2, 3, 4 are located the number of the stars is not less than 4. It follows by the Dirichlet principle that at least one of the stars should be substituted by an integer which is not less than 8 (the number of the integers 5, 6 and 7 is exactly equal to 3). This ends the proof.

Problem 18. The floor of a class room is coloured arbitrarily in black and white. Prove that there exist 2 monochromatic points on the floor such that the distance between them is exactly 1 m.
Solution: Take an equilateral triangle on the floor with side 1 m. It follows by the Dirichlet principle that at least 2 of the vertices are monochromatic and we are done.

Problem 19. The floor of a class room is coloured arbitrarily in black and white. Prove that there exist 3 collinear monochromatic points on the floor such that one of them lies in the middle of the segment connecting the others.
Solution: Consider 5 collinear points *A*, *B*, *C*, *D* and *E* on the floor such that *B* and *D* are monochromatic (say white), *AB* = *BD* = *DE* and *BC* = *CD*. If at least one of the points *A*, *C* or *E* is white, than such a point together with *B* and *D* solves the problem. If all the three are black, then they solve the problem.

Problem 20. Given 7 segments such that their lengths are greater than 10 cm and smaller than 1 m. Prove that a triangle could be constructed by 3 of them.
Solution: Arrange the segments according to their size: $a_1 \leq a_2 \leq ... \leq a_7$. Use that if $a_k \leq a_{k+1} \leq a_{k+2}$, then the segments a_k, a_{k+1} and a_{k+2} are sides of a triangle if and only if $a_k + a_{k+1} > a_{k+2}$. Assume that no 3 of the segments form a triangle. Then $a_k + a_{k+1} \leq a_{k+2}$ for all *k* = 1, 2, 3, 4 and 5. It follows from $a_1 \geq 10$ and $a_2 \geq 10$ that $a_3 \geq 20, a_4 \geq 30, a_5 \geq 50, a_6 \geq 80$ and $a_7 \geq 130$, which is a contradiction.

Problem 21. Prove that at least 2 of the sides of an arbitrary convex polyhedron are with one and the same number of vertices.
Solution: The sides of a polyhedron are polygons. Assume that each couple of sides of a polyhedron is with different number of vertices. Consider the side with the maximal number of vertices and denote this number by *n*. It follows that the side under consideration has *n* edges and consequently it is adjacent to *n* sides of the polyhedron.

The assumption implies that each of the adjacent sides has 3, 4, 5, ..., $n - 2$ or $n - 1$ vertices. Since the number of the integers from 3 to $n - 1$ is equal to $n - 3$ it follows by the Dirichlet principle that at least 2 of the adjacent sides are with the same number of vertices. This is a contradiction.

Problem 22. 9 vertices of a regular icosagon (20-gon according to the Greek "ico" which means twenty) are coloured in red. Prove that there exists an isosceles triangle with red vertices.
Solution: Enumerate the successive vertices of the polygon by the integers from 1 to 20 clock-wisely. The vertices with numbers 1, 5, 9, 13 and 17; with numbers 2, 6, 10, 14 and 18; with numbers 3, 7, 11, 15 and 19; with numbers 4, 8, 12, 16 and 20 determine 4 regular pentagons. Since the coloured vertices are 9 it follows by the Dirichlet principle that at least 3 of them belong to one and same pentagon. On the other hand any 3 vertices of a pentagon form an isosceles triangle and we are done.

Problem 23. Several arcs of a circle are coloured in red (some of the arcs could overlap). If the sum of the lengths of the coloured arcs is less than the half of the circle's length, then prove that there exists a diameter of the circle with uncoloured end points.
Solution: Also colour the arcs in red which are symmetric to the given ones with respect to the center of the circle. Then the sum of the lengths of all coloured arcs does not exceed the length of the circle. Consequently a point exists on the circle which is not coloured. This point together with the antipodal one solves the problem.

Problem 24. More of the half surface of a spherical planet is covered by land while the other part is covered by water. Prove that at least one pair of diametrically opposite points lie on the land.
Solution: Denote the set of the points on the land of the planet by A. Let B be the set of the points which are diametrically opposite to the points of A. Since A covers the greater part of the planet surface it follows that B also covers the greater part of the planet surface. If there is a point in B which lies on the land then we are done. If such a point does not exist it follows that the greater part of the planet surface is water and this contradicts to the condition of the problem.

Problem 25. A 6 x 6 square is divided into 36 closed unit squares. Find the maximal number of unit squares which could be coloured in blue in a way that no blue unit squares have common points.
Solution: Divide the given square into 9 squares 2 x 2 which are the drawers. It is not possible to have 2 blue unit squares in such a square 2 x 2. It follows by the Dirichlet principle that the searched number is equal to 9 at most. The given example is a realization of 9 blue unit squares which are marked by **X**:

X		**X**		**X**	
X		**X**		**X**	

X		**X**		**X**	

Problem 26. Prove that it is not possible to cover an equilateral triangle with side *a* by 5 equilateral triangles with sides smaller than $\frac{1}{2}a$.

Solution: Let ΔABC be the equilateral triangle with side *a* and *M*, *N* and *P* be the midpoints of the sides *AB*, *BC* and *AC*, respectively. Assume that ΔABC is covered by 5 equilateral triangles with sides smaller than $\frac{1}{2}a$. It follows by the Dirichlet principle that at least 2 of the 6 points *A*, *B*, *C*, *M*, *N* and *P* are located in one of the 5 triangles. Then, the length of the side of that triangle is not less than $\frac{1}{2}a$, which is a contradiction.

Problem 27. Given a square with side 1 and 101 points are taken in it in such a way that no 3 of them are collinear. Prove that 3 of the points form a triangle with area not greater than 0,01.

Solution: Divide the square into 50 equal rectangles. It could be done by dividing the side of the square into 10 equal parts and the adjacent side into 5 equal parts. The horizontal and vertical lines through the points of division divide the square into 50 equal rectangles with dimensions 0,2 x 0,1. It follows by the Dirichlet principle that at least 3 of the 101 points fall into one and the same rectangle. Now use the following fact: if a triangle is situated in a parallelogram, then the area of the triangle does not exceed the half area of the parallelogram. In the concrete situation the area of the rectangle is equal to 0,02 and consequently the area of the triangle does not exceed 0,01. Thus we are done.

What is used in the previous problem is a particular case (for a rectangle) of the following:

Lemma. If a triangle is situated in a parallelogram, then the area of the triangle does not exceed the half area of the parallelogram.

Proof: If one of the sides of the triangle lies on a side of the parallelogram, then it does not exceed the corresponding side of the parallelogram. At the same time the altitude of the triangle to the side under consideration does not exceed the altitude of the parallelogram. Now the assertion is obvious. If one of the sides of the triangle is parallel to a side of the parallelogram, then obviously the reasoning can be reduced to the previous one by taking a smaller parallelogram. It remains the case when no side of the triangle lies on a side of the parallelogram and no side of the triangle is parallel to a side of the parallelogram. In this case take lines through the vertices of the triangle which are parallel to one and the same side of the parallelogram. One of the lines is between the other two and it divides the parallelogram into 2 new parallelograms. Also, it divides the triangle into 2 new triangles. Already, each of the new triangles has a side on a side of a new parallelogram. Further, apply the reasoning from the beginning for this case.

Problem 28. A convex 10-gon is situated in a square with side 1. Prove that there exists a triangle with vertices among the vertices of the 10-gon and with area which does not exceed 0,08.

Solution: Denote the 10-gon by $A_1A_2...A_{10}$. Since it is situated in a square, then its perimeter does not exceed the perimeter of the square, i.e. it does not exceed 4. Take the sum of the lengths of two consecutive sides of the 10-gon and consider all such sums, i.e. consider the following 10 numbers: $A_1A_2 + A_2A_3$, $A_2A_3 + A_3A_4$, …, $A_{10}A_1 + A_1A_2$. The sum of the 10 numbers is equal to the double perimeter of the 10-gon, i.e. it does not exceed 8. It follows by the Dirichlet principle that at least one of the 10 numbers does not exceed 0,8. Let *n* be such that $A_nA_{n+1} + A_{n+1}A_{n+2} \le 0{,}8$. Further, use that the area of a triangle does not exceed the semi-product of any 2 sides of the triangle. In fact the area is equal to the semi-product of a side and the altitude to it but the altitude does not exceed each of the adjacent sides. Consequently, a positive integer *n* exists between 1 and 8 such that the area of the triangle $A_nA_{n+1}A_{n+2}$ does not exceed $\frac{1}{2}ab$, where $A_nA_{n+1} = a$ and $A_{n+1}A_{n+2} = b$. On the other hand $a + b \le 0{,}8$. Since $(a+b)^2 \ge 4ab$, then $\frac{1}{2}ab \le \frac{(a+b)^2}{8} \le \frac{0{,}8 \times 0{,}8}{8} = 0{,}08$ and we are done.

Problem 29. A 4 x 4 chess board is covered by 13 domino plates with dimensions 1 x 2 in a way that each of the two halves of a domino plate covers exactly one field of the chess board. Prove that one of the domino plates could be removed while the chess board would remain still covered.
Solution: There are two cases of placing the domino plates. If one of the 13 domino plates is such that its two halves are overlapped by the halves of other domino plates, then it could be removed obviously and we are done. In the second case the contrary situation is present, i.e. at least one half of each domino plate is not overlapped by the half of any other domino plate. Consequently there are 13 halves which cover fields of the chess board and are not overlapped by other halves. In this manner the covered fields are exactly 13. The remaining 3 fields (4 x 4 – 13 = 3) are covered by the other 13 halves. It follows by the Dirichlet principle that one of these 3 fields is covered by 5 halves at least, i.e. at least 5 domino plates participate in the covering of the field under consideration. At the same time each field could be covered by domino plates in 4 different ways: the free half of the domino plate is to the right, to the left, upwards or downwards. Thus, at least 2 domino plates overlap fully and one of them could be removed.

Problem 30. The plane is divided into unit squares (in this case we say that an integer net is defined on the plane). The vertices of the unit squares are called knots. Prove that for any 5 knots there exist 2 of them such that the middle of the segment between them is a knot, too.
Solution: Consider the coordinates of the 5 knots. All of them are integers and their remainders modulo 2 are equal to 0 or 1. It follows that there are 4 possibilities for the remainders of the 5 knots: (0,0), (0,1), (1,0) and (1,1). The Dirichlet principle implies that at least 2 knots *M* = (*a*,*b*) and *N* = (*c*,*d*) coincide modulo 2, i.e. the integers *a* and *c*, as well as the integers *b* and *d*, have the same remainders modulo 2. It follows that the numbers $\frac{a+c}{2}$ and $\frac{b+d}{2}$ are integer. Consequently the middle $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$ of the segment *MN* is a knot.

Problem 31. Given 2 equal 16-gons and 7 vertices are coloured in red for each of them. Prove that the 16-gons could be overlapped in such way that at least 4 red vertices of the first 16-gon coincide with 4 red vertices of the other.

Solution: Overlapping the 16-gons in a way that each vertex of one of the 16-gons coincides with a vertex of the other, there exist 16 different rotations (the full rotation included) of one of the 16-gons keeping the property of vertex coincidence. If after each rotation the number of the pairs of coinciding red vertices is equal to 3 at most, then all pairs are 16 x 3 = 48 at most. On the other hand the number of the possible pairs is equal to 7 x 7 = 49. It follows that there is a rotation after which the number of the pairs of coinciding red vertices is 4 at least.

MATHEMATICAL GAMES

Svetoslav Bilchev, Emiliya Velikova
UNION OF BULGARIAN MATHEMATICIANS

Section 1. MATHEMATICAL JOKES

In the following *mathematical games – mathematical jokes* the final result depends only on the initial conditions of the game but not on the strategies of the players.

1.1. Problem. Let *m* points are given in the plane. Two players consequently connect any two disjoint points with an arc no intersecting any other already existing arc. The winner is the player who takes the last move.

Solution. If $m = 2$, the first player is the winner.

Let $m > 2$. In this case we have to use the following:

Euler's Theorem. Let *m* points are given in the plane and *n* pairs nonintersecting arcs connecting some two points and not passing through the remaining $m-2$ points. Let the so given plane is divided into l regions. If from any point is possible to go to any other point along the given arcs it follows $m-n+l=2$.

At the end of the game ($m>2$) we obtain a map every two vertexes of which are connected by the chain of arcs. Every side of the map is bounded by three arcs, i.e. $2n=3l$. From the *Euler's theorem* it follows that the number *n* of the arcs in such map is equal to $3(m-2)$. But the number of the arcs on the map is equal to the number of the moves in the game.

Thus, if *m* is an odd (*even*) number and $m>2$ the winner is the first (*second*) player.

Section 2. SYMMETRY

Here we consider mathematical games in which the winner takes fundamentally using of idea of symmetry.

2.1. Problem. In a heap there are *1992* stones. Two players take part in the following game: everyone from them for one move can take any amount of the stones which is a devisor of the amount of the stones which was taken by the previous player's move. The first move of the player can be any amount of the stones but not all of them. The winner is the player who takes the last stone.

Solution. The winner is the first player. He takes *8* stones in his first move and the heap remains with $1984 = 64 \times 31$ stones. After that the first player repeats every move of the second player. The second player has the right to take only *1, 2, 4* or *8* stones. At the same time *16* divides *1984*. Then the number of the moves will be exactly even number (without the first move). Hence, the first player will take the last move.

2.2. Problem. On a circle n points numbered by $1, 2, \ldots, n$ are given. Two players consequently joint with a horde any pair points with the same parity. Every horde has not any common point (even the edge point) with the existing already hordes. The looser is the player who has not any further move.

Solution. The second player is the winner if $n = 4k + 2$. The first player is the winner in all other cases.

Let us consider that the given points are the vertices of a regular *n–gon*.

Case 1. Let $n = 4k$. The first player creates a diameter of the circle with his first move and after that on every move of the second player he answers with a symmetric move with respect to this diameter.

Case 2. Let $n = 4k + 2$. Then the second player on every move of the first player answers with the symmetric horde with respect to the centre of the circle. The main point here is that the diametrically opposite points have different parity.

Case 3. Let $n = 4k + 1$. Then on the circle there are two next odd points numbered by 1 and n. With the first move the first player connects the points numbered by 1 and *3* and so he transforms the given game to the same game but with $n = 4(k-1) + 2$ in which the started player is the looser.

Case 4. Let $n = 4k + 3$. With the first move the first player connects the points numbered by $2k + 1$ and $2k + 3$. Then if we try mentally move the remaining $4k + 1$

points so that they form a regular $(4k+1)$–*gon* the diametrically opposite points will be with opposite parity and the second player is the looser in this game.

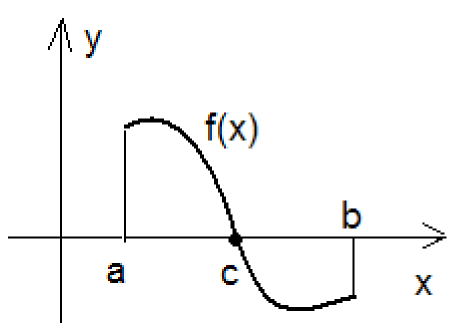
2.3. Problem. A field with $m \times n$ unit squares is given. For one move any of two players have possibility to paint any unit square. It is not allowed to paint more than two unit squares in any combination of four squares lying on the crossing of any two rows and any two columns. The looser is the player who has not more moves. Who will win in a right play? Consider the following examples of fields: i) 4×6 ; ii) 5×5; iii) 4×7; iv) $m \times n$.

Hint. In the case *ii)* and in any other case when **m** and **n** are odd numbers the winner is the first player. He has to follow the symmetrical strategy: at the beginning he paints the central square and after that he paints the central symmetrical square to this square which has been painted from the second player before. When on the field there is at least one "even" side it is necessary to rotate the field to have even number of rows. On every move of the first player the second player has to answer by painting another square on the same column. Two rows where the players are painting now are closed. The first player has to paint further in a new column, second player answers with painting a square in the same column and after that two more rows are closed. But the number of the rows is even and soon the first player will finish his moves. The winner is the second player.

Section 3. GAMES WITH POLYNOMIALS

For this type of games is characteristic some special choice of coefficients of the polynomials for obtaining the necessary conditions of their roots. For this purpose is necessary to use the following theorem.

Theorem. If the function $f(x)$ is continuous on the interval $[a,b]$ and $f(a).f(b) < 0$ then the point c exists such that $f(c) = 0$.



3.1. Problem. On the blackboard is written the equation

$$f(x)=x^3+a_1x^2+a_2x+a_3=0 .$$

Two players play the following game – the first one puts some real number instead of one of the coefficients of the equation, the second one do the same with the other of the coefficients. At the end the first player changes the last coefficient in the same manner. The first player is the winner if the equation has three different real roots. In any other case the winner is the second player.

Solution. The first player can win the game with the following strategy. He takes the first move with changing of the coefficient a_2 so that $1+a_2<0$. Further on after the second player's move the first player takes the last remaining coefficient such that $a_1+a_3=0$. Then

$$f(1)=1+a_1+a_2+a_3<0 \text{ and}$$
$$f(-1)=-1+a_1-a_2+a_3>0 .$$

By using *The Theorem* we conclude that the function $f(x)$ has three real roots respectively on the intervals $(-\infty,-1)$, $(-1,1)$, $(1,\infty)$.

3.2. Problem. On the blackboard is written the equation

$$*\,x^2+*x+*=0$$

every star in which means some real coefficient. Two players are playing the following game. The first player takes any three real numbers and the second player puts them instead of the stars as he wants. The first player is the winner if the so obtained equation has two different rational roots.

Solution. The first player is always the winner if he takes three different integers *a, b, c* as coefficients such that $a+b+c=0$. Then the equation will have the roots: $x_1=1, x_2=\frac{c}{a}$, $(c\neq a)$.

3.3. Problem. On the blackboard is written the equation:

$$f(x)=x^3+a_1x^2+a_2x+a_3=0.$$

Two players consequently change the coefficients with integers not equal to zero. The first player is the winner if $f(x)$ has at least two different integer roots. The second player is the winner in any other case.

Solution. The winner is the first player with first move $a_2=-1$ and with second move (third move in the game) - the opposite number of the second player's number. So the function $f(x)$ can be expressed in the two following ways:

$$x^3+ax^2-x-a=(x+a)(x^2-1) \text{ or}$$
$$x^3-ax^2-x+a=(x-a)(x^2-1).$$

Hence so obtained function $f(x)$ has two different integer roots: $+1$ and -1 .

3.4. Problem. The equation

$$f(x)=x^4+a_1x^3+a_2x^2+a_3x+a_4=0$$

is given. The game is the following: the first player puts instead of one of the coefficients any non zero integer and then the second player chooses for the other three coefficients some non zero integers. If $f(x)$ has at least two different integer roots the winner is the second player. In any other case the winner is the first player.

Solution. The winner is the first player if he takes $a_4=-1$. Then the equation

$$x^4+a_1x^3+a_2x^2+a_3x-1=0$$

can have integer roots only equal to (+1) or (-1). But if they are roots of $f(x)$ then

$$a_1+a_2+a_3=0 \quad \text{and}$$
$$-a_1+a_2-a_3=0\ ,$$

from where $a_2=0$ which is impossible by the rules of the game.

3.5. Problem. On the blackboard is written the equation

$$f(x)=x^3+*x^2+*x+*=0.$$

The first player says some real number and the second player puts this number instead of any star. After that the first player offer another real number which the second player puts instead one of the two remaining stars. At the end the first player puts any real number instead of the last star. The first player is the winner if the so obtained equation has three different integer roots. In any other case the winner is the second player.

Solution. The winner is the first player. His first number has to be *0*.

<u>*Case 1.*</u> If the second player puts *0* instead of the last star then $f(x)=x^3+*x^2+*x$. After that the first player chooses the number *2* and at the end – the number: ***-3*** . Then

$$f(x)=x(x-1)(x-2) \quad \text{or}$$
$$f(x)=x(x-1)(x+3)\ .$$

<u>*Case 2.*</u> If the second player puts ***0*** instead of the first star it follows $f(x)=x^3+bx+c$. Then the first player says the number: $-(3.4.5.)^2$. If the second player puts this number instead of ***b*** , then $c=0$, and if instead of ***c*** , the first player puts $b=3^24^2-3^25^2-4^25^2$. Hence respectively

$$f(x)=x(x+3.4.5)(x-3.4.5) \quad \text{and}$$
$$f(x)=\left(x+3^2\right)\left(x+4^2\right)\left(x-5^2\right).$$

<u>*Case 3.*</u> If the second player puts ***0*** on the second star it follows $f(x)=x^3+ax^2+c$. Now the first player says the number: 6^27^3 and after that he puts $a=-7^2$ or $c=-6^87^6$. Hence respectively

$$f(x)=(x+2.7)(x-3.7)(x-6.7) \quad \text{and}$$
$$f(x)=\left(x-2.6^27^2\right)\left(x+3.6^27^2\right)\left(x+6^37^2\right)\ .$$

3.6. Problem. Two players consequently change the coefficients of the polynomial $P(x) = a_0 + a_1 x + a_2 x^2 ... + a_{1000} x^{1000}$ with integers. The winner is the first player if the obtained polynomial has one and the same reminder after dividing it by ***6*** for every integer ***x.*** In any other case the winner is the second player.

Solution. Let we note that for every integer k the number ***6*** divides the numbers $k^3 - k = (k-1)k(k+1)$ and $k^4 - k^2 = k.(k-1)k(k+1)$. Therefore for dividing of the polynomial

$$Q(x) = ax + bx^2 + cx^3 + dx^4$$

by ***6*** for every integer k is sufficient to be done

$$a + c = 0 \quad \text{and} \quad b + d = 0 .$$

Let we unite in the polynomial $P(x)$ the terms from the first power till to the fourth power, from the fifth power till to the eighth power and so on. At the end we obtain

$$P(x) = a_0 + \sum_{k=0}^{k=249} x^{4k} f_k(x) ,$$

somewhere the functions $f_k(x)$ have the same structure as $Q(x)$.

So the first player can take the following strategy: to put a_0 equal to the desired reminder and after that on every choosing the coefficient from $f_k(x)$ of the second player the first player answers with choosing the coefficient so that to fulfill one of the correspondent equalities: $a + c = 0$ and $b + d = 0$.

Hence the obtained polynomial $P(x)$ with integer coefficients will have the chosen reminder a_0 on every integer x after dividing it by *6*.

3.7. Problem. The polynomial

$$P(x) = x^{10} + *x^9 + *x^8 + ... + *x^2 + *x + 1$$

is given. Two players consequently change the stars of the polynomial with integers (in general - 9 moves). The winner is the first player if the obtained polynomial has not real roots. If the polynomial has at least one real root the winner is the second player. Is it possible the second player to be the winner in any play of the first player?

Solution. The answer is: Yes, he has a winning strategy in any play of the first player!

It is necessary to change 9 stars – 5 ahead of the odd powers and 4 ahead of the even powers. If the first player changes a star ahead of an *even* (**odd**) power with some coefficient then the second player has to change a star ahead of an *odd* (**even**) power with some coefficient. So after seven moves two stars remains ahead of the powers x^k and x^l , where at least one the numbers k and l is an odd number and the second player is on a move. Let after seven moves we have

$$P(x)=Q(x)+\alpha x^k+\beta x^l \quad .$$

There are two cases:

i) k - even number, l - odd number. Then:

$$P(1)=Q(1)+\alpha+\beta, P(-1)=Q(-1)+\alpha-\beta, P(1)+P(-1)=Q(1)+Q(-1)+2\alpha.$$

The second player has to choose $\alpha=-\frac{1}{2}[Q(1)+Q(-1)]$. Therefore independently of the last move of the first player we will have: $P(1)+P(-1)=0$. Hence, or $P(1)=P(-1)=0$ and the polynomial $P(x)$ has even two real roots: (+1) and (-1), either $P(1)=-P(-1)$, i.e. $P(1).P(-1)<0$ and the polynomial $P(x)$ will have at least one real root on the interval $[-1,1]$, (see the figure at the beginning of this part).

ii) k and l - odd numbers. Then:

$$P(-1)=Q(-1)-\alpha-\beta \quad , \quad P(2)=Q(2)+\alpha.2^k+\beta.2^l \ ,$$

from where: $2^l.P(-1)+P(2)=2^l.Q(-1)+Q(2)+\left(2^k-2^l\right)\alpha$.

The second player has to choose $\alpha=-\dfrac{2^l.Q(-1)+Q(2)}{2^k-2^l}$. Therefore independently of the last move of the first player we will have: $2^l.P(-1)+P(2)=0$. Hence, or $P(-1)=P(2)=0$ and the polynomial $P(x)$ has even two real roots: (-1) and (+2), either $2^l.P(-1)=-P(2)$, i.e. $P(-1).P(2)<0$ and the polynomial $P(x)$ will have at least one real root on the interval $[-1,2]$, (see the figure at the beginning of this part).

Section 4. MINIMAX

In this part we will consider games in which the payoff of every player is variable with different number values dependent from the moves of the players and every player want to increase his payoff.

The games will be with two players which sum of the payoffs is a constant value independent from the players. The interests of the players are directly opposite because when the payoff of one of the players increases then the payoff of the second player decreases.

4.1. Problem. A boy and a girl divide among each other 10 suits on the following way: the boy divides the suits on two heaps and the girl takes one of them. How many suits are possible to be taken by the boy and the girl?

Solution. Everyone will take exactly 5 suits. Really, the boy will not divide the heap on different number of suits because the girl will take the biggest part. Then the boy makes as smaller as is possible the maximal payoff of the girl. Such kind of strategy we name **“minimax”** strategy.

4.2. Problem. The numbers $1, 2, 3, \ldots, 20$ are written on the blackboard. Two players consequently put ahead of every number the sign $(+)$ or the sign $(-)$**.** The sign is possible to put ahead of every free number. The first player wants to obtain at the end the smallest by module sum but the second one wants to make this sum as bigger as it is possible. What value can obtain the second player?

Solution. The biggest sum for the second player is 30.

Let us consider the strategy of the second player for obtaining the biggest sum. We divide the numbers on ten pairs: $(1, 2), (3, 4), \ldots, (19, 20)$.

The first player on every his move wants to put the sign $(+)$ ahead of the biggest of the numbers in every pair, the second player will answer with the opposite sign ahead of the second number of the pair.

Only in the case when the first player puts some sign ahead of the number from the last pair then the second player have to put the same sign ahead of the second number of this pair.

It is evident that the module of the so obtained sum is not less than

$$19 + 20 - 1 - 1 - \ldots - 1 = 30.$$

Now we will prove that the first player have possibility to bound the second player’s sum on not more than 30**.** He has to put ahead of the biggest of the remaining numbers the opposite sign of the sign of the sum for those moment (if the sum is equal to zero the first player puts the sign $(+)$).

Let us consider an example of the game and let the *k-move* is the last one move when the sum changes its sign (including the moves when the sum is equal to zero). For the first $k-1$ moves obviously the numbers $20, 19, 18, \ldots, 20-(k-1)$ have been used. Then the maximal by module sum which is possible to be obtained after the *k-move* is equal to

$$20-(k-1)+20-k = 41-2k.$$

For everyone of the following $10-k$ moves the sum goes done at least with ***1*** because the first player every time subtracts from the module of the sum the biggest of the remaining numbers m but the second player can add to it not more than $m-1$. Hence, the final result can be not more than

$$(41-2k)-(10-k) = 31-k \le 30 \ .$$

Section 5. WINING STRATEGIES

In every game given below one of the players has winning strategy.

5.1. Problem. Let we have a table with dimension 3×3 and *9* cards of unit dimension. On every card is written one of the numbers:

$$a_1 < a_2 < \ldots < a_9.$$

Two players consequently put one of non used cards on a free cage of the table. After using of all cards the first player adds the six numbers on the cards lying on the upper and on the down rows of the table and the second player adds the six numbers of the cards lying on the left and on right columns. The winner is the player with the biggest sum.

Solution. The winner is the first player or the game is equal.

If $a_1 + a_9 > a_2 + a_8$ the first player puts a_9 on the cage 1 and for the second move he puts a_2 or a_1 on one of the cages *2* or *3*.

If $a_1 + a_9 < a_2 + a_8$ the first player puts a_1 on the cage 2 and for the second move he puts a_9 or a_8 on one of the cages *1* or *4*.

If $a_1 + a_9 = a_2 + a_8$ the first player can use one of the given up strategies.

	1	
2		3
	4	

5.2. Problem. A convex polyhedron with $n \geq 5$ faces is given. From every vertex of this polyhedron exactly three edges erected. Two players consequently draw its names on one of the free faces. The winner is those of the players who first is drawing his name on three faces passing through one vertex.

Solution. The winner is the first player.

It is necessary to prove at the beginning that there exists a face which is not a triangle. Let all faces are triangles. Then the polyhedron has $\frac{3n}{2}$ edges because three edges erected from every vertex and every edge belongs at the same time to two vertexes. By using *the Euler's Theorem* $(V + F - E = 2)$ we obtain $n + n - \frac{3n}{2} = 2$, i.e. $n = 4$, which is in a contradiction with the conditions of the game.

So we have a face, let us call it A_1, which is not a triangle. The first player have to put his name on A_1. With the second own move the first player has to use the face A_2, next to the face A_1, which has common edges with two free faces A_3 and A_4 lying next to the face A_1 as well (it is possible because the second player can use only one face next to the face A_1). At the end with the third own move the first player can use one of the faces A_3 or A_4 which is not used by the second player. So the winner is the first player.

Section 6. THEORY OF NUMBERS

6.1. Problem. Two players consequently write one $2p$*-digits number* by using the digits *1, 2, 3, 4, 5.* The first player writes the first digit, the second player writes the second digit, the first player writes the third digit and so on…. If the obtained number is divisible by *9* the winner is the second player. In the opposite case the winner is the first player.

Solution. Let the first player writes the digits - $a_1, a_2, \ldots, a_p$, the second one – the digits - $b_1, b_2, \ldots, b_p$ and $S = a_1 + a_2 + \ldots + a_p + b_1 + b_2 + \ldots + b_p$.

<u>*Case 1.*</u> If $p = 3m$ the second player is the winner with the strategy: $b_i = 6 - a_i$, then the sum

$$S = (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_p + b_p) = 6p = 18m$$

is divisible by *9*, i.e. the obtained number is divisible by *9*.

<u>*Case 2.*</u> If $p = 3m + 1$ or $p = 3m + 2$ the first player is the winner with the strategy: $a_1 = 3$ and after that $a_i = 6 - b_{i-1}$, then the sum

$$S = a_1 + (a_2 + b_1) + (a_3 + b_2) + \ldots + (a_p + b_{p-1}) + b_p = 3 + 6(p-1) + b_p$$

If $p = 3m + 1$ then the sum $S = 18m + (3 + b_p)$ is not divisible by *9* because $3 + b_p$ is between *4* and *8*.

Analogically, for $p = 3m + 2$, then the sum $S = 18m + 9 + b_p$ is not divisible by *9* because b_p is between *1* and *5*.

6.2. Problem. Two players write consequently some $2p$*-digits number* by using only the digits *6, 7, 8, 9*. The first player writes the first digit, the second player writes the second digit, the first player writes the third digit and so on…. If the obtained number is divisible by *9* the winner is the second player. In the opposite case the winner is the first player.

Solution.

<u>*Case 1.*</u> If $p = 3n$, the second player is the winner. After every move of the first player the second player writes a digit which sum with the last written by the first player digit gives the reminder *6* after dividing it by *9* (after *6* the second player writes *9*, after $7 - 8$, after $8 - 7$, after $9 - 6$).

<u>*Case 2.*</u> Let $p = 3n + 1$. The winner is the first player. The first his move has to be any digit without *9.* Further after every move of the second player the first player writes a digit which sum with the last written by the second player digit gives the reminder *6* after dividing it by *9*. After such strategy of the first player the sum of all digits without the first and the last ones is divisible by *9*. But the first digit is different from *9* and, hence, the sum of all digits is not divisible by *9*.

<u>*Case 3.*</u> If $p = 3n + 2$ the first player is the winner with the following strategy: the first move of the first player is the digit *9*. Then on every move of the second player without of the last one the first player answered on the same manner as in the <u>*Case 2.*</u> If the second player's move which is before the last one is a digit not equal to *9* then the

first player puts immediately after that the digit *9*. With this strategy the sum of all digits without the first one and the last three digits is divisible by *9*. Among those "extraordinary" four digits there is at least one which is different from *9*. Hence, the sum of those four digits is not divisible by *9* and then the same is with the sum of all digits.

6.3. Problem. Several players sitting at a round table are numbered clockwise. The first one has one euro more than the second, the second one has one euro more than the third, and so on every player has one euro more than the next one. The first player gives one euro to the second player, the second gives two euro to the third player and so on every player gives one euro more than he has received. The game continues until it is possible. At the end of the game it turns out that one of the players has money 4 times as much as one of his neighbors. How many are the players and how much money had the "poorest" of them at the beginning?

Solution. Let n be the number of the players and let the "poorest" of them (for example, the $n-th$ player) has x euro at the beginning. From the rules of the game after the first "round" the players have (according to their numbering)

$$x+2n-2, \quad x+n-3,\ldots, \quad x+1, \quad x, \quad x-1 \text{ euro.}$$

Hence the game continues x "rounds" until the last $n-th$ player finishes his x euro. Then at the end the players have respectively

$$x+(x+1)(n-1), \quad n-2, \quad n-3,\ldots, \quad 2, \quad 1, \quad 0 \text{ euro.}$$

Since $0, \ 1, \ 2,\ldots, \ n-2$ are consecutive natural numbers then only the first player can have money 4 times as much as his neighbor. But his neighbors are the $2-nd$ player and the $n-th$ player. Since the last player has *0* euro at the end of the game then from the conditions we get the equation

$$x+(x+1)(n-1)=4(n-2), \qquad i.e. \qquad x=\frac{3n-7}{n}=3-\frac{7}{n} .$$

Therefore n divides 7 and it can be only *1* or *7* . But when $n=1$ we get $x<0$ which is impossible. Hence $n=7$ and $x=2,$ i.e. *7* players are around the table and the "poorest" of them has *2* euro at the beginning.

Remark. The above arguments were done under the assumption that $x \neq 0$, but one can easily see that if $x=0$ the situation described in the problem statement would not be possible.

6.4. Problem. A number of k motorcyclists, $(k \geq 1)$, have to travel from A to B. The first day the motorcyclist M_i, $(i=1,2,\ldots,k)$, covered $\frac{1}{n_i}$ of the whole distance, where n_i is some positive integer. The second day he covered $\frac{1}{m_i}$ of the remaining distance, where m_i is some positive integer. The third day he covered $\frac{1}{n_i}$ of the distance left after the second day, the fourth day he covered $\frac{1}{m_i}$ of the way left after the third day. The pairs (m_i, n_i) and (m_j, n_j) of natural integers are different if $i \neq j$,$(i\ , j = 1,2,\ldots,k)$. At the end of the fourth day it turns out that every motorcyclist M_i covered exactly $\frac{3}{4}$ of the distance between A and B.

i) Find the biggest possible positive integer k ;

ii) Which motorcyclist M_i will be the winner in a race with the above given conditions if the product $m_i n_i$ of his numbers m_i, n_i have to be possible the biggest one?

Solution. Let S denote the distance between A and B. The first day the motorcyclist M_i covered $\frac{1}{n_i}S$ and the remaining distance was

$$S - \frac{1}{n_i}S = \left(1 - \frac{1}{n_i}\right)S \quad .$$

The second day he covered $\frac{1}{m_i}\left(1 - \frac{1}{n_i}\right)S$ and the remaining distance was

$$\left(1 - \frac{1}{n_i}\right)S - \frac{1}{m_i}\left(1 - \frac{1}{n_i}\right)S = \left(1 - \frac{1}{n_i}\right)\left(1 - \frac{1}{m_i}\right)S \quad .$$

Similarly we find out that after the fourth day the remaining distance was

$$\left(1 - \frac{1}{n_i}\right)^2\left(1 - \frac{1}{m_i}\right)^2 S \quad .$$

Thus we get the following equation

$$\left(1 - \frac{1}{n_i}\right)^2\left(1 - \frac{1}{m_i}\right)^2 S = \frac{1}{4}S \text{ , i.e. } \left[\frac{(m_i - 1)(n_i - 1)}{m_i n_i}\right]^2 - \left(\frac{1}{2}\right)^2 = 0\text{, from where}$$

$$\frac{(m_i - 1)(n_i - 1)}{m_i n_i} = +\frac{1}{2} \quad \left(or = -\frac{1}{2}\right) .$$

Since $m_i \geq 1, n_i \geq 1$ then it follows

$$\frac{(m_i - 1)(n_i - 1)}{m_i n_i} = +\frac{1}{2} \quad \text{, i.e.} \quad m_i = 2 + \frac{2}{n_i - 2} \quad .$$

But m_i is an integer then $n_i - 2$ must divide 2. And using that n_i is a positive integers we get that n_i is either 3 or 4. Then m_i is either 4 or 3 respectively.
Hence, we obtain only two solutions: the motorcyclist M_1 with $(m_1, n_1) = (3, 4)$ and the motorcyclist M_2 with $(m_2, n_2) = (4, 3)$. Thus

i) the biggest possible positive integer k is $k=2$.

ii) we have not a winner in such race because $m_1 n_1 = m_2 n_2 = 3.4 = 4.3 = 12$.

6.5. Problem. The numbers $1, 2, 3, \ldots, 27$ are given. Two players consequently cross out a number while two numbers remain. If their sum is divisible by *5* the winner is the first player. In other case – the winner is the second player. Who will be the winner in a right play?

Hint. It will be more convenient to consider not the given numbers but their reminders with respect to *5.* Those reminders are: *5* zeros, *5* fours, *5* threes, *6* units and *6* twos. The first player is the winner. The first move of the first player is to cross out the number ***1***. After that he has to cross out such reminders which sum with the corresponding crossing reminders from the second player is equal to *5*. More precisely: *i)* if the moves of the second player are *1, 2, 3, 4* then the first player moves are *4, 3, 2, 1*; *ii)* if the move of the second player is *0* then the first player move is *2* and after that on every move *0* of the second player he will answer with move *0*. Using this strategy at the end two reminders will remain: *0, 0* or *2, 3* or *1, 4.*

6.6. Problem. On a circle n boxes are given. One of the boxes is full with two stones – white and black. The other boxes are empty. Two players consequently move the stones – the first moves clockwise the white stone through one or two boxes, the second moves the black stone on the opposite side through one or two boxes as well. The winner is this player who puts his stone into the box with the stone of the other player. Who will be the winner in a right play? Consider the cases: i) $n=13$; ii) $n=14$; iii) $n=15$; iv) n is any natural number.

Hint. It is necessary to point that: *a)* if four empty boxes are between the stones this is *"a trap"* – the player who has to move is the looser; *b)* if three empty boxes are between the stones this is *"a pass"* – the stones will pass on the following move through each other without *"fighting".* Use $n = 5k + s, \quad (s = 0, 1, 2, 3, 4)$.

6.7. Problem. The price of a good is n euro. Two directors of stores are playing. Every one of them with his own move increases the price of the good with $m\%$, where m is a natural number, $m \in [1, 100)$, and at the same time with integer number of euro. The looser is the player which has not any more moves in some moment. Which of the directors is the looser? Consider the following cases: i) $n=1000$; ii) $n=880$; iii) $n=600$; iv) $n=2^k$; v) n is any natural number.

Answer. The winner is the second player if $n = m.2^k.5^s$, where $m = 10p + q,\ (q = 1, 3, 7, 9\ ),\ \ p$ is any natural number, and *3* divides the natural numbers k and s. In all other cases the winner is the first player.

Section 7. APPENDIX

7.1. Problem. A kangaroo is jumping within the angle $x \geq 0, y \geq 0$ of the coordinate plane Oxy in the following way: from the point (x, y) the kangaroo can jump to the point $(x+1, y-1)$ or to the point $(x-5, y+7)$ but it is impossible for the kangaroo to jump to the points a coordinate of which is negative. From which initial points (x, y) the kangaroo can not get into the point which distance from the centre O of the coordinate plane Oxy is more than 1000 units. Draw the set T of the all such points (x, y) and compute the area F of the set T.

Answer. $F = 15$. The set $T(x, y)$ is: the column-like triangle without the all points on the "staircase":

$$T(x, y) \equiv \{(x = k + \alpha,\ 0 \leq y < 5 - k,\ k = 0,1,2,3,4;\ 0 \leq \alpha \leq 1) \setminus$$
$$\setminus (x = k,\ 5 - k < y < 6 - k,\ k = 1,2,3,4,5)\} \equiv \{[x] + [y] \leq 4, x \geq 0, y \geq 0\},$$

where $[x]$ is the bracket function, i.e. in the above considered case if $x = n, \ldots$ is a nonnegative decimal number, then $[x] = n$ is a natural number.

7.2. Problem. An automat is working on the following way: after putting inside *5* cents the automat products by *3* the given to it number and after putting inside *2* cents the automat adds *4* to the given to it number.

i) What minimal sum of cents is necessary to use for obtaining the number $n = 1979$ from the number *1* with the help of the given automat?

ii) What is the minimal sum of cents if $n = 2005$?

Hint. It is necessary to think inversely – from n to *1*!

i) The minimal necessary sum of cents is: $37 = 5.5 + 6.2$. See the series

$$1979 \Leftarrow 1975 \Leftarrow 1971 \Leftarrow 657 \Leftarrow 219 \Leftarrow 73 \Leftarrow 69 \Leftarrow 23 \Leftarrow 19 \Leftarrow 15 \Leftarrow 5 \Leftarrow 1$$

7.3 Problem. Choose 2^n different natural numbers between the numbers *1* and 3^n inclusively so that the average value of every two chosen numbers is not within the set of the chosen numbers.

Hint. Use the mathematical induction on n. The set of choosing numbers is:

$$a_1 = 1,\ a_2 = 2,\ a_3 = 7,\ a_4 = 8,\ \ldots,\ a_{2^{n-1}},\ a_{2^{n-1}+1} = a_1 + 2.3^{n-1},$$

$$a_{2^{n-1}+2} = a_2 + 2.3^{n-1},\ \ldots,\ a_{2^n} = a_{2^{n-1}} + 2.3^{n-1}\ .$$

GEOMETRY IN THE PLANE

Emiliya Velikova, Svetoslav Bilchev
UNION OF BULGARIAN MATHEMATICIANS

Section 1. INTRODUCTION

This chapter offers several general methods for solving problems in plane geometry and is designed for leading teachers of mathematics and their talented students.
The chapter consists of twelve sections: introduction, triangle, polygons, circles, extreme problems, loci, construction problems, transformations (rotation, similarity and inversion), special theorems, metric problems, geometric inequalities, applications of vectors in geometry.
All sections include material which starts from topics covered at the ordinary school curriculum but continues beyond – for example several problems are solved by using of complex numbers, vectors, transformations, algebraic and geometric inequalities etc.
Almost all of the 41 given problems are with full solutions.

Section 2. TRIANGLE

Problem 2.1. Find three distinct isosceles triangles with integer sides such that each triangle's area is numerically equal to six times its perimeter.

Solution. Let the sides of such a triangle be (a,b,b). Then six times the perimeter of the triangle is equal to $6a+12b$ and the area of the triangle, by the Heron's formula, is equal to $\sqrt{\left(b+\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(b-\frac{a}{2}\right)}$. It follows that

$$6a+12b=\sqrt{\left(b+\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(b-\frac{a}{2}\right)}\ ,$$

$$144\left(b+\frac{a}{2}\right)^2=\left(b+\frac{a}{2}\right)\left(\frac{a}{2}\right)^2\left(b-\frac{a}{2}\right)\ ,$$

$$144\left(b+\frac{a}{2}\right)=\left(\frac{a}{2}\right)^2\left(b-\frac{a}{2}\right),\ \ \frac{2b+a}{2b-a}=\left(\frac{a}{24}\right)^2\ .$$

Trying multiples of 24 for a leads to the following three solutions: $a=48,\ b=40$; $a=72,\ b=45$; $a=120,\ b=65$. These have areas of 972, 768 and 1500, respectively, all of which are six times their perimeters.

Problem 2.2. Let G be the centroid of a triangle ABC and let D be the midpoint of side BC. Suppose triangle BDG is equilateral with side length 1. Determine the lengths of the sides $AB,\ BC,\ CA$ of ABC.

Solution. Side BC: The point D is the midpoint of side BC, so $DC = 1$ and $BC = 2$.

Side AB: The point G is the centroid of the triangle ABC, so $AG = 2GD$ and so $AG = 2,\ AD = 3$. The triangle BDG is an equilateral triangle, so $\angle GDB = 60^0$. Then, by the Law of Cosines for the triangle ABD,

$$AB^2 = AD^2 + BD^2 - 2 \cdot AD \cdot BD \cdot cos\left(\angle GDB\right) = 1^2 + 3^2 - 2 \cdot 1 \cdot 3 \cdot \frac{1}{2} = 7,$$

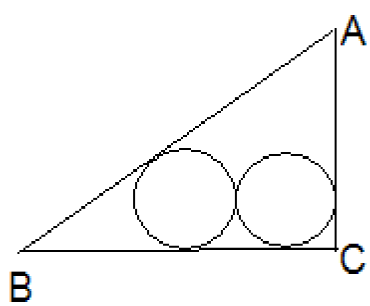
hence $AB = \sqrt{7}$.

Side AC: Angles $\angle GDC$ and $\angle GDB$ are supplementary, so $\angle GDC = 120^0$. Then, by the Law of Cosines for the triangle ACD,

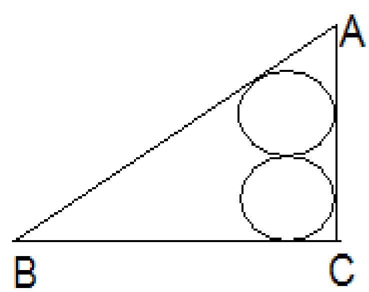
$$AC^2 = AD^2 + CD^2 - 2 \cdot AD \cdot CD \cdot cos\left(\angle GDC\right) = 1^2 + 3^2 - 2 \cdot 1 \cdot 3 \cdot \left(-\frac{1}{2}\right) = 13,$$

or $AC = \sqrt{13}$.

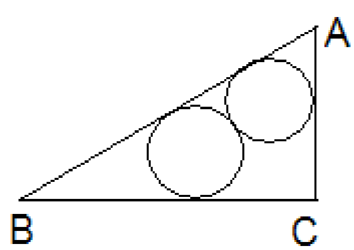
Problem 2.3. Two circles with equal radius can fit tightly inside a right angled triangle ABC, which has sides $AB = 13,\ BC = 12,\ CA = 5$, in the three positions illustrated below. Determine the radii of the circles in each case.



Case 1



Case 2



Case 3

Answer. Case 1. $\frac{3}{2}$**; Case 2.** $\frac{10}{9}$**; Case 3.** $\frac{26}{17}$.

Problem 2.4. Let P be a point on side BC of a triangle ABC. A line through P parallel to AB cuts AC at E, and a line through P parallel to AC cuts AB at F. If the area of triangle ABC is equal to 1, prove that the area of one of BPF, CPE and $AEPF$ is not less than $\frac{4}{9}$.

Solution. Let $BC=1,\ BP=r(0,1)$. Then $CP=1-r,\ F_{\Delta BPF}=r^2$ ($F_{\Delta BPF}$ - denotes the area of the triangle *BPF*) and $F_{\Delta CPE}=(1-r)^2$.

If $r\geq\frac{2}{3}$, then $F_{BPF}=r^2\geq\frac{4}{9}$. If $r\leq\frac{1}{3}$, then $F_{\Delta CPE}=(1-r)^2\geq\frac{4}{9}$.

Finally, suppose $r\in\left(\frac{1}{3},\frac{2}{3}\right)$. Then $F_{AEPF}=1-r^2-(1-r)^2=2r-2r^2$.

The obtained function is a parabola which opens downwards, so that it's minimum value is at one of the end points. If $r=\frac{1}{3}$ or $r=\frac{2}{3}$, then $F_{AEPF}=2r-2r^2=\frac{4}{9}$.

Hence, $F_{AEPF}>\frac{4}{9}$, or the area of one of the triangles BPF, CPE and $AEPF$ is not less than $\frac{4}{9}$.

Problem 2.5. A triangle ABC with sides $|AB|=12,\ |BC|=13,\ |CA|=15$ is given. A point M on the side AC is such that the radii of the circles inscribed in the triangles ABM and BCM are equal. Find the ratio $|AM|:|MC|$.

Solution. Let $|AM|:|MC|=k$. If the radii of the circles inscribed in the triangles ABM and BCM are equal then the ratio k of their areas is equal to the ratio of their perimeters.
Hence,

$$|BM|=\frac{13k-12}{1-k}.$$

In particular, it follows from this equality that

$$\frac{12}{13}<k<1.$$

Write down now the laws of cosine for triangles ABM and BCM (with respect to the angles BMA and BMC). After that eliminate the cosines of the angles from those equations. We obtain the following equation for k:

$$69k^3-112k^2+44k=0$$

with roots 0, $\frac{2}{3}$ and $\frac{22}{23}$.

Finally, we get $k=\dfrac{22}{23}$ taking into account the limitations for k .

Remark: It is also possible to apply Steward's Great Theorem:

$$BM^2=\frac{CM\cdot AB^2+AM\cdot BC^2}{AC}-AM\cdot CM$$

for obtaining the equation for k .

Problem 2.6. Let ABC be a triangle whose incenter is ℓ. Consider the circle Ω tangent to the sides CA,CB respectively at D,E and interior tangent to the circumcircle. Prove that ℓ is the midpoint of the segment DE.

Solution. It is obvious that if we succeed in proving that ℓ lies on the segment DE, then the problem is solved because triangle DCE is isosceles, and CI is an angle bisector and hence a median of DCE.

Let us denote by x,y the lengths of the segments BE,AD, respectively. By Cauchy's theorem applied to the points A,B,C and the circle Ω we obtain

$$xb+ya=(a-x)c.$$

But $CE=CD$. Thus $a-x=b-y$, i.e. $y=b-a+x$.

Solving the above system we obtain:

$$x=\frac{a(s-b)}{s} \text{ and } y=\frac{b(s-a)}{s},$$

where *s* is the semi perimeter of the triangle ABC.

The fact that ℓ lies on DE shows is equivalent, by the transversal theorem, to the equality

$$AC'\cdot\frac{BE}{EC}+BC'\cdot\frac{AD}{DC}=AB\cdot\frac{C'I}{IC}$$

where $C'=CI\cap AB$.

We know from the bisector theorem that

$$\frac{C'I}{IC}=\frac{c}{a+b}$$

so the previous equality is equivalent to

$$\frac{cb}{b+a}\cdot\frac{x}{a-x}+\frac{ca}{a+b}\cdot\frac{y}{b-y}=\frac{c^2}{a+b} \text{ , i.e. } \frac{bx}{a-x}+\frac{ay}{b-y}=c \text{ or}$$

$$\frac{b\cdot\frac{a(s-b)}{s}}{a-\frac{a(s-b)}{s}}+\frac{a\cdot\frac{b(s-a)}{s}}{b-\frac{b(s-a)}{s}}=c \qquad \text{or} \qquad \text{finally}$$

$$\frac{ba(s-b)}{sa-as+ab}+\frac{ab(s-a)}{sb-bs+ba}=c\ .$$

But last equality is obviously true.

Problem 2.7. Let ABC be a triangle and M,N - the midpoints of sides BC,AC, respectively. If the orthocenter of the triangle ABC and the centroid of the triangle AMN coincide, determine the angles of the triangle ABC.

Solution. We will solve this problem by using complex numbers.
Let us take the circumcenter O of the triangle ABC as the origin of the coordinate plane and denote by a,b,c the complex numbers that are the affixes of the points A,B,C, respectively.
The orthocenter of the triangle ABC corresponds to the complex number

$$h = a + b + c \quad .$$

The centroid of the triangle AMN corresponds to the complex number

$$g = \frac{1}{3}\left(a + \frac{b+c}{2} + \frac{c+a}{2}\right) = \frac{3a+b+2c}{6}.$$

From $g = h$ we get $3a + 5b + 4c = 0$. Now, without loss of generality we may assume $a = 1$ and consequently $|b| = |c| = 1$ (since each is equal to the radius 1 of the circle).
The previous equality becomes $3 + 5b + 4c = 0$ and taking conjugates one also has

$$3 + 5\bar{b} + 4\bar{c} = 0 \quad \text{or} \quad 3 + \frac{5}{b} + \frac{4}{c} = 0.$$

Solving for b and c the system given by the two equalities one obtains either

$$c = i,\ b = -\frac{3}{5} - i\frac{4}{5} \quad \text{or} \quad c = -i,\ b = -\frac{3}{5} + i\frac{4}{5}.$$

The obtained triangles are congruent since they are symmetrical with respect to the real axis. By standard computation we obtain

$$B = \frac{\pi}{4},\ \tan A = 3,\ \tan C = 2.$$

Problem 2.8. A triangle ABC with orthocenter H, circumcenter and circumradius R is given. Let D,E,F be the reflection of points A,B,C along BC,CA,AB, respectively. Show that D,E,F are collinear if and only of $OH = 2R$.

Solution. Let G be the centroid of the triangle ABC, and A',B',C' be the midpoints of BC,CA,AB, respectively.
Let $A''B''C''$ be the triangle for which A,B,C are the midpoints of $B''C'',C''A''$, $A''B''$ respectively. Then G is the centroid and H is the circumcenter of the triangle $A''B''C''$.
Let D',E',F' denote the projections of O on the lines $B''C'',C''A'',A''B''$ respectively. Consider the similarity h with center G and ratio $-\frac{1}{2}$. It maps A,B,C,A'',B'',C'' into A',B',C',A,B,C, respectively.
Note that $A'D' \perp BC$ which implies $\frac{AD}{A'D'} = \frac{2}{1} = \frac{GA}{GA'}$ and $\angle DAG = \angle D'A'G$.

We conclude that $h(D)=D'$ and, similarly, $h(E)=E'$, $h(F)=F'$. Thus, D,E,F are collinear if and only if D',E',F' are collinear. But, D',E',F' are the projections of O on the sides $B''C'',C''A'',A''B''$ respectively. By Simpson's theorem, they are collinear if and only if O lies on the circumcircle of the triangle $A''B''C''$. Since the circumradius of $A''B''C''$ is $2R$, the point O lies on the circumcircle if and only if $OH=2R$.

Problem 2.9. The medians AD, BE, CF of the triangle ABC intersect at the point G. Six small triangles, each with a vertex at G, are formed. We draw the circles inscribed in the triangles AFG, BDG, CDG. Prove that if these three circles all are congruent, then the triangle ABC is equilateral.

Hint. From triangles BDG and CDG easily follows that $BG=CG$, so $BE=CF$ and hence $AB=AC$, i.e. $c=b$. Then from the triangles AGF and BGD, using the fact that they have equal areas, congruent inscribed circles and congruent medians $m_a, m_b, m_c=m_b$ we can obtain:

$$\frac{F_{\Delta AGF}}{\frac{b}{2}+\frac{2}{3}m_a+\frac{1}{3}m_b}=\frac{F_{\Delta BGD}}{\frac{a}{2}+\frac{2}{3}m_b+\frac{1}{3}m_a}, \text{ i.e. } 3(b-a)=2(m_b-m_a),$$

This is equivalent to the following equation of third degree:

$$\left(\frac{b}{a}-1\right)^2\left(\frac{b}{a}-\frac{1}{2}\right)=0$$

and, because of $b+c=2b>a$, we obtain $b=a$, i.e. $a=b=c$.

Section 3. POLYGONS

Problem 3.1. Suppose that in a convex quadrilateral $ABCD$, the areas of the triangles ABD,BCD and ABC are in proportion $3:4:1$. If a line through B cuts AC at M and CD at N in such a way that $AM:AC=CN:CD$, prove that M and N are the midpoints of AC and CD respectively.

Solution. Let $AM:AC=CN:CD=r$ and $F_{\Delta ABC}=1$. Then

$$F_{\Delta ABD}=3,\ F_{\Delta BCD}=4 \text{ and } F_{\Delta ACD}=3+4-1=6 \text{ (Figure 1).}$$

Hence

$$F_{\Delta ABM}=r,\ F_{\Delta BCN}=4r \text{ and } F_{\Delta ACN}=6r.$$

Now,

$$F_{\Delta BCM}=F_{\Delta ABC}-F_{\Delta ABM}=1-r,\quad F_{\Delta CNM}=F_{\Delta BCN}-F_{\Delta BCM}=5r-1 \text{ and}$$
$$F_{\Delta AMN}=F_{\Delta ACN}-F_{\Delta CMN}=r+1.$$

From $\frac{r+1}{6r}=\frac{F_{\Delta AMN}}{F_{\Delta ACN}}=\frac{AM}{AC}=r$ we obtain the quadratic equation

$$0=6r^2-r-1=(2r-1)(3r+1).$$

Since $r > 0$, we must have $r = \frac{1}{2}$, which leads to the desired result.

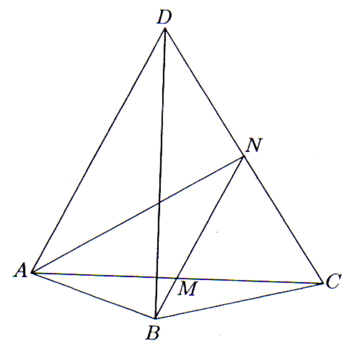


Figure 1

Problem 3.2. Let $ABCDEF$ be an inscribed hexagon in which $|AB| = |CD| = |EF| = R$, where R is the radius of the circumscribed circle and O *is* its center. Prove that the points of pair wise intersections of the circles circumscribed about the triangles BOC, DOE, FOA distinct from the point O, form an equilateral triangle of side R.

Solution. It is obvious that the triangles AOB, COD, EOF are equilateral. Denote:

$$\angle BOC = 2\alpha,\ \angle DOE = 2\beta,\ \angle FOA = 2\gamma.$$

Let K, M, L be, respectively, the intersection points of the circles circumscribed about the triangles BOC and AOF, BOC and DOE, AOF and DOE. The point K lies inside the triangle AOB, and

$\angle BKO = 180^0 - \angle BCO = 90^0 + \alpha,\quad \angle AKO = 90^0 + \gamma$ and, since

$\alpha + \beta + \gamma = \dfrac{2\pi - 3\cdot\frac{\pi}{3}}{2} = 90^0$, it follows that $\angle AKB = 90^0 + \beta$.

Similarly, L lies inside the triangle FOE, and

$$\angle OLF = 90^0 + \gamma,\quad \angle OLE = 90^0 + \beta,\ \angle FLE = 90^0 + \alpha.$$

Because of the equilateral triangles *AOB* and *DOF* it follows that $|OL| = |AK|$ and

$$\angle KOL = 2\gamma + \angle KOA + \angle LOF = 2\gamma + \angle KOA + \angle KAO = 90^0 + \gamma = \angle AKO.$$

Thus, the triangles KOL, AKO are congruent, that is, $|KL| = |AO| = R$.

Similarly, $|LM| = |MK| = R$.

Problem 3.3. Prove that the interior of a convex pentagon $ABCDE$ having all sides of equal length cannot be entirely covered by the open discs having the sides of the pentagon as diameters.

Solution. Let us denote by $2R$ the side length of $ABCDE$. It follows directly from the Pigeonhole Principle that there are two consecutive angles of the pentagon greater than

$\frac{2\cdot 3\pi}{2\cdot 5}=\frac{3\pi}{5}>\frac{\pi}{3}$, i.e. greater than 60^0. Suppose that these angles are $\angle EAB, \angle ABC$. It follows that BE and AC are greater than $2R$.

Let M be the midpoint of segment EC. The point M is on the semicircle of diameter DE and DC $(DM \perp CE)$, therefore it lies in their exterior. We shall prove that M also lies in the exterior of the semicircle of diameter AE. Indeed,

$$MF=\frac{AC}{2}>R,$$

where F is the midpoint of the segment AE. The same follows for the semicircle of diameter BC.

All we have left to prove is that M is at the exterior of the semicircle of diameter AB. Suppose otherwise, which means that $\angle AMB>90^0$. Then AB is the greatest side of the triangle AMB, thus $AM<2R$. But from

$$EM=\frac{1}{2}EC<\frac{1}{2}(ED+DC)=2R$$

it follows that $EA=2R>EM$, $EA>AM$, and thus $\angle EMA>60^0$.

In the same way $\angle CMB>60^0$ and therefore

$$180^0<210^0<\angle EMA+\angle AMB+\angle CMB,$$

which is a contradiction.

Problem 3.4. Let $ABCDE$ be a cyclic pentagon inscribed in a circle of center O and suppose that

$$\angle B=120^0,\ \angle C=120^0,\ \angle D=130^0,\ \angle E=100^0.$$

Show that the diagonals BD and CE meet at a point belonging to the diameter AO.

Solution. We shall use complex numbers. By standard computations we find that, on the circumscribed circle, the sides of the pentagon are supported by the following arcs:

$$arcAB=80^0,\ arcBC=40^0,\ arcCD=80^0,\ arcDE=20^0,\ arcEA=140^0.$$

It is then natural to consider all these measures as multiples of 20^0 which corresponds to the 18th – primitive root of unity, say

$$\varpi=cos\frac{2\pi}{18}+i\,sin\frac{2\pi}{18}.$$

We thus assign, to each vertex, starting form $A=1$ the corresponding root of unity:

$$B=\varpi^4, C=\varpi^6, D=\varpi^{10}, E=\varpi^{11}.$$

We shall use the following properties of ϖ:

$$\varpi^{18}=1\ \ or\ \ \bar{\varpi}^k=\varpi^{18-k};\ \ \varpi^9=-1\ \ or\ \ \varpi^6-\varpi^3+1=0.$$

We need to prove that the complex number corresponding to the common point of lines BD and CE (Figure 2) is in fact a real number.

The equation of line BD is

(1) $$\begin{vmatrix} z & \overline{z} & 1 \\ \varpi^4 & \overline{\varpi}^4 & 1 \\ \varpi^{10} & \overline{\varpi}^{10} & 1 \end{vmatrix} = 0 \ ,$$

and the equation of the line CE is

(2) $$\begin{vmatrix} z & \overline{z} & 1 \\ \varpi^6 & \overline{\varpi}^6 & 1 \\ \varpi^{11} & \overline{\varpi}^{11} & 1 \end{vmatrix} = 0 \ .$$

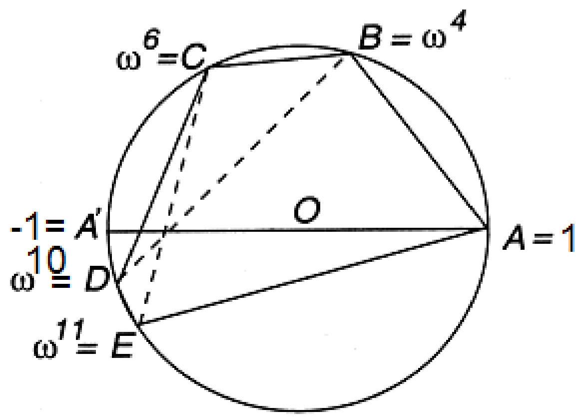


Figure 2

Equation (1) can be written as follows:

$$z\left(\varpi^{14} - \varpi^8\right) - \overline{z}\left(\varpi^4 - \varpi^{10}\right) + \left(\varpi^{12} - \varpi^6\right) = 0$$

or $$z\varpi^8\left(\varpi^6 - 1\right) + \overline{z}\varpi^4\left(\varpi^6 - 1\right) + \varpi^6\left(\varpi^6 - 1\right) = 0 .$$

Using the properties of ω we derive at the following simplified version of (1):

(3) $$z\varpi^4 + \overline{z} + \varpi^2 = 0.$$

In the same way equation (2) becomes

(4) $$z\varpi + \overline{z} - \varpi^3\left(\varpi^4 - 1\right) = 0 \ .$$

From (3) and (4) we obtain the following expression for z

$$z = \frac{-\varpi^7 + \varpi^3 - \varpi^2}{\varpi^4 - \varpi} = \frac{-\varpi^6 + \varpi^2 - \varpi}{\varpi^6} = -1 + \frac{\varpi - 1}{\varpi^5} \ .$$

To prove that z is real, it will suffice to prove that it coincides with its conjugate. It is easy to see that

$$\frac{\varpi - 1}{\varpi^5} = \frac{\overline{\varpi} - 1}{\overline{\varpi}^5}$$

is equivalent to

$$\overline{\varpi}^4 - \overline{\varpi}^5 = \varpi^4 - \varpi^5,$$

that is

$$\varpi^{14}-\varpi^{13}=\varpi^{4}-\varpi^{5}\text{, i.e. } \varpi^{10}-\varpi^{9}=1-\varpi\text{, } \left(\varpi^{9}+1\right)\left(\varpi-1\right)=0$$

which is true due to the relation $\varpi^{9}=-1$.

Problem 3.5. Let $P_1P_2...P_n$ be a convex polygon in a plane. We assume that for any arbitrary choice of vertices P_i, P_j there exists a vertex of the polygon from which the segment $\left[P_iP_j\right]$ can be seen at an angle of 60^0. Show that $n=3$.

Solution. Let P_j, P_k be vertices such that the side P_jP_k has minimal length and let P_i be the vertex which satisfies the condition $\angle P_jP_iP_k=60^0$. Then the triangle $P_jP_iP_k$ is equilateral (Prove that!). Let us denote it by ΔABC .

In the same way, taking a side P_rP_s of maximal length and the vertex P_t from which $\angle P_rP_tP_s=60^0$, one obtains an equilateral triangle, denoted by $\Delta A_1B_1C_1$. We shall prove that $AB=A_1B_1$. This ends the problem, because the polygon is convex and its vertices are in the domains D_A, D_B, D_C (Figure 3).

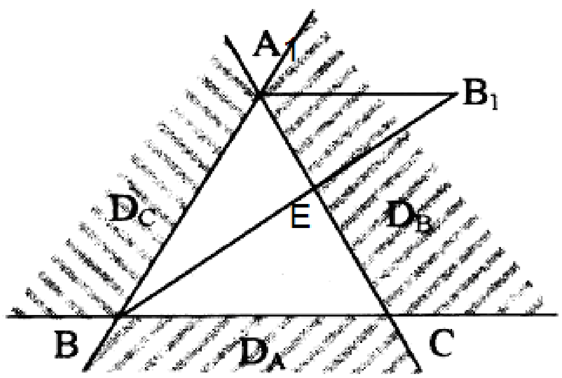


Figure 3

We distinguish the following two cases:

***Case 1*.** The triangles ΔABC and $\Delta A_1B_1C_1$ have a common vertex. Say $A=A_1$. Then B_1 and C_1 are not both in the domain D_A because in such a case $\angle B_1A_1C_1<60^0$. Assume $B_1\in D_B$. Then the segments B_1B and AC have a common point E. The following inequality holds:

$$AB_1+BC<AC+BB_1\ ,$$

because $AB_1<AE+B_1E$ and $BC<BE+CE$.

It follows that $AB_1<BB_1$, which contradicts the maximal length of $AB_1=A_1B_1$. The conclusion is that B_1 is not in D_B nor in D_C. Then B_1 is one of the points B or C.

Case 2. The triangles ΔABC and $\Delta A_1B_1C_1$ have no common vertex. If two of the points A_1, B_1, C_1 are in the same domain, say $A_1, B_1\in D_C$ (Figure 4) if follows that

$\angle B_1CA_1 < 60^0$ and then $max(CB_1, CA_1) > A_1B_1$. This is a contradiction.

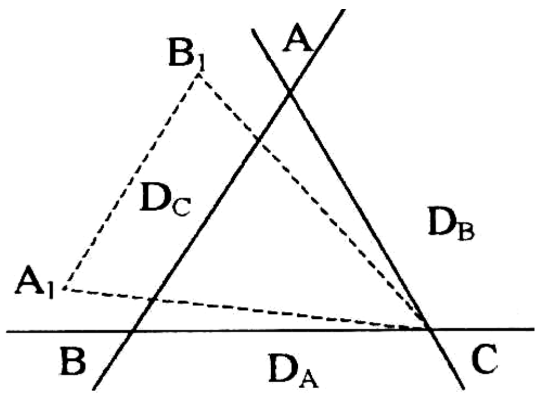


Figure 4

If the points A_1, B_1, C_1 are in distinct domain D_A, D_B, D_C it follows that A, B, C are exterior points of the triangle $\Delta A_1B_1C_1$, since the polygon is convex (Figure 5).

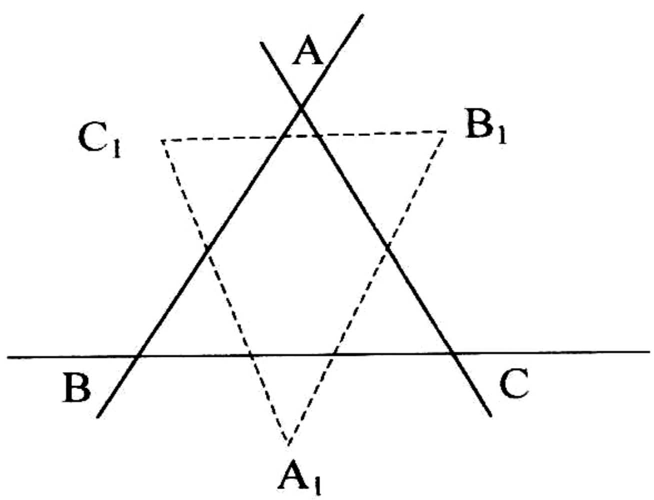


Figure 5

Because BC has minimal length, it follows that $\angle BA_1C \leq 60^0$ and then $\angle C_1A_1B_1 < 60^0$.
This is a contradiction.
We conclude that case 2 cannot occur.

Section 4. CIRCLES

Problem 4.1. Let AB be the diameter of a semi-circle and T a point on the extension of BA, with $AT < \frac{1}{4}AB$. Assume that a line ℓ passes through T and is perpendicular to AB and that lines through two distinct points M and N on the semicircle and perpendicular to ℓ intersect it at P and Q, respectively. If $MP = AM$ and $NQ = AN$, prove that $AM + AN = AB$.

Solution. In Figure 6, we project M and N onto C and D on AB. Since the triangles CAM and MAB are similar, we have $AM^2 = AC.AB$. Similarly, $AN^2 = AD.AB$. Hence

$$AB.CD = AB(AD - AC) = AN^2 - AM^2 = (AN + AM)(AN - AM) =$$
$$= (AN + AM)(NQ - MP) = (AN + AM).CD$$

or $AM + AN = AB$.

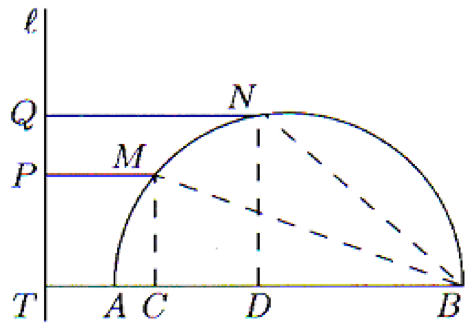


Figure 6

Problem 4.2. Suppose two circles $C_1(O_1)$ and $C_2(O_2)$ with distinct radii meet at points A and B and suppose that the tangent from A to C_1 intersects the tangent from B to C_2 at a point M. Show that both circles are seen from the point M by the same angle.

Solution. We have to prove that $2\angle O_1MA = 2\angle O_2BM$ (Figure 7), which is equivalent to

(5) $$\frac{O_1A}{AM} = \frac{O_2B}{BM}.$$

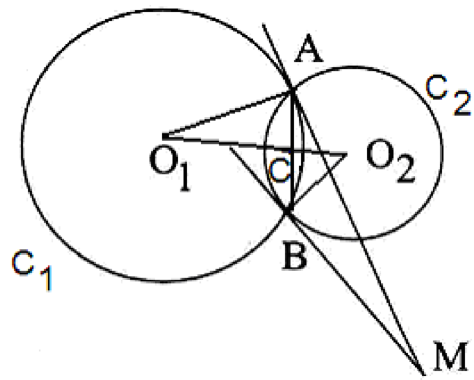


Figure 7

Let O_1O_2 intersect AB in C. The length of the common chord AB is equal to

$$AB = 2 \cdot O_1A \cdot \sin \angle AO_1C = 2 \cdot O_1A \cdot \sin \angle BAM$$

and similarly

$$AB = 2 \cdot O_2B \cdot \sin \angle BO_2C = 2 \cdot O_2B \cdot \sin \angle ABM ,$$

hence

(6) $$\frac{O_1A}{\sin(\angle ABM)} = \frac{O_2B}{\sin(\angle BAM)} .$$

By the Sine Theorem in the triangle *ABM* we derive that

(7) $$\frac{MA}{\sin(\angle ABM)} = \frac{MB}{\sin(\angle BAM)} .$$

By dividing (6) and (7) we obtain (5), as desired.

Problem 4.3. Suppose that three circles of a plane whose centers are the points A,B,C, respectively, are each tangent to a line **Δ** and pairwise externally tangent to one another. Prove that the triangle ABC has an obtuse angle and find all possible values of this angle.

***Solution*.** Denote the radii of the three circles by a,b,c, respectively. Let also A',B',C' be the projections of the centers A,B,C on the line **Δ** (Figure 8). Suppose $c \le a < b$. Then

$$A'B' = \sqrt{(a+b)^2 - (a-b)^2} = 2\sqrt{ab} ,\ B'C' = 2\sqrt{bc} ,\ A'C' = 2\sqrt{ac} .$$

From the equality

$$A'B' = A'C' + C'B'$$

it follows that

$$\sqrt{ab} = \sqrt{ac} + \sqrt{bc} ,$$

which is equivalent to the equality $c = \dfrac{ab}{\left(\sqrt{a} + \sqrt{b}\right)^2}$.

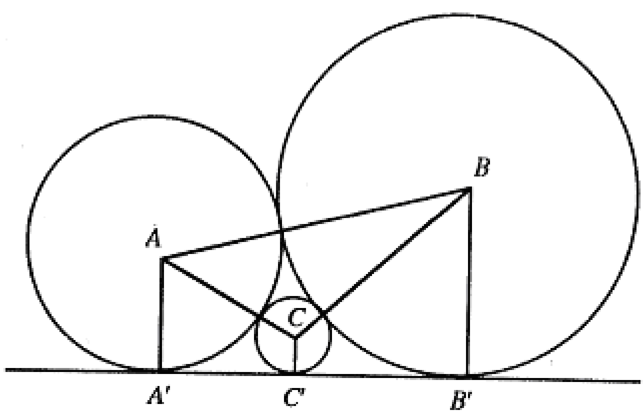


Figure 8

From the Cosine Rule applied to the triangle ABC we deduce that

(8) $$\cos C = \frac{c(a+b+c)-ab}{(a+c)(b+c)}.$$

It is easy to see that $\angle C$ is obtuse because of the following equivalences:

$$\cos C < 0 \Leftrightarrow c(a+b+c) < ab = c\left(\sqrt{a}+\sqrt{b}\right)^2 \Leftrightarrow$$
$$a+b+c < a+b+2\sqrt{ab} \Leftrightarrow c < 2\sqrt{ab} \Leftrightarrow c^2 < 4ab \quad ,$$

which is obvious from $c \le a$, $c < b$.

The measure of $\angle C$ is given by (8), which can be rewritten as

$$\cos C = 1 - \frac{2ab}{(a+c)(b+c)}.$$

Equivalently, we obtain

(9) $$\sin^2 \frac{C}{2} = \frac{ab}{(a+c)(b+c)}.$$

Because $\frac{\pi}{2} < C < \pi$, it follows that $\frac{\pi}{4} < \frac{C}{2} < \frac{\pi}{2}$. Hence, it is sufficient to find the maximum of the function $\sin^2 \frac{C}{2}$, given by (9).

The formula (9) can be written in the form

$$\sin^2 \frac{C}{2} = \frac{1}{\frac{a+c}{a}\cdot\frac{b+c}{b}} = \frac{1}{\left(1+\frac{c}{a}\right)\left(1+\frac{c}{b}\right)}$$

and the original problem reduces to finding the maximum of the product

$$P = \left(1+\frac{c}{a}\right)\left(1+\frac{c}{b}\right).$$

Denote $\frac{\sqrt{c}}{\sqrt{a}} = x$ and $\frac{\sqrt{c}}{\sqrt{b}} = y$.

Then $P = \left(1+x^2\right)\left(1+y^2\right)$, with the supplementary conditions $x+y=1,\ x,y \ge 0$.

Using Calculus the problem is simple but we shall solve it by elementary methods. Set $xy = p$. Then

$$P = 1+x^2+y^2+x^2y^2 = 2-2xy+x^2y^2 = p^2-2p+2,$$ where

$$p = xy \le \left(\frac{x+y}{2}\right)^2 \le \frac{1}{4}.$$

The quadratic function $P = p^2-2p+2$ is a decreasing function on the interval $\left(0,\frac{1}{4}\right]$.

Hence, the function takes minimal value for $p = \frac{1}{4}$ and then $P = \frac{25}{16}$.

The conclusion is that for $x=y$ or $a=b$ we obtain $max(\angle C)=2arcsin\dfrac{16}{25}$.

Therefore, the possible values for $\angle C$ are: $\angle C \in \left(\dfrac{\pi}{2},\, 2arcsin\dfrac{16}{25}\right]$.

Problem 4.4. Consider a triangle ABC with circumcircle $k(O)$ and a point D on the side BC. Let $k_1(L)$ be a circle tangent to $k(O)$, AD and BD and let circle $k_2(L)$ be a circle tangent to $k(O)$, AD and DC. Show that $k_1(K)$ *and* $k_2(L)$ are tangent if and only if

$$\angle BAD = \angle CAD.$$

Solution. Let E be the intersection point of the line AD and the circle $k(O)$ (Figure 9).

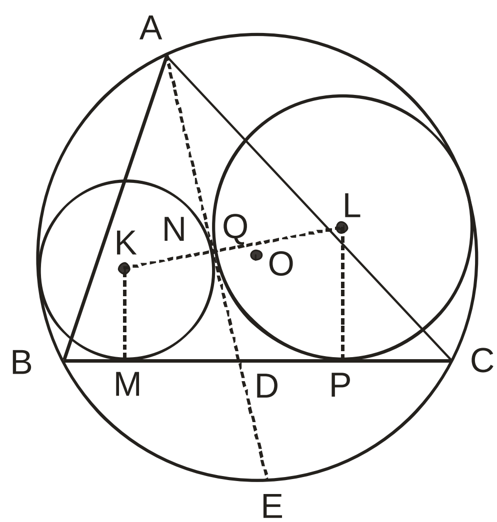


Figure 9

Let M and N be the tangent points of the circle $k_1(K)$ with BD and AD, respectively, and let P and Q be the tangent points of the circle $k_2(L)$ with DC and AD, respectively as well.

We apply the **Cauchy's theorem** for the circles B, C, E and $k_1(K)$, where the points B, C and E are considered as degenerate circles. In this way we obtain

$$BE \cdot CM + CE \cdot BM = BC \cdot EN.$$

By applying ones again the Cauchy's theorem for the circles B, C, E and $k_2(L)$ we obtain

$$BE \cdot CP + CE \cdot BP = BC \cdot EQ.$$

The circles $k_1(K)$ *and* $k_2(L)$ are tangent if N and Q coincide, i.e. $EN \equiv EQ$. By the above relations, this condition is equivalent to

$$BE(CM - CP) = CE(BP - BM) \Leftrightarrow (BE - CE)MP = 0.$$

It follows $BE = CE$. This condition is equivalent to $\angle BAE = \angle CAE$.

Remark. This solution is very easy to apply for solving the next similar problem:

Problem 4.5. The circles $k(O),\ k_1(K),\ k_2(L)$ are related to each other as follows: the circles $k_1,\ k_2$ are externally tangent to one another at a point N and both these circles are internally tangent to the circle k. Points A,B,C are located on the circle k as follows: BC is a direct common tangent to the pair of circles $k_1,\ k_2$ and line NA is the transverse common tangent at N to $k_1,\ k_2$, with N and A lying on the same side of the line BC. Prove that N is the in center of the triangle ABC.

Section 5. EXTREME PROBLEMS

Problem 5.1. In a square of side 6 the points A,B,C,D are given such that the distance between any two of the four points is at least 5. Prove that A,B,C,D form a convex quadrilateral and its area is greater than 21.

Solution. First of all we observe that no angle formed with three of the four points can be greater or equal to 120^0, because otherwise if we suppose that $\angle ABC \geq 120^0$, then from $AB \geq 5$ and $BC \geq 5$ we deduce $AC \geq 5\sqrt{3} > 6\sqrt{2}$, which is a contradiction.
Therefore if the quadrilateral $ABCD$ is not convex then one of the four points lies inside the triangle formed by the other three points. Suppose that $D \in int[ABC]$. But then one of the angles $\angle ADB, \angle BDC, \angle CDA$ would be, by the Pigeonhole principle, greater or equal to 120^0, contradiction. Thus, $ABCD$ is a convex quadrilateral.
Now because each angle of the triangle ABC is smaller than 120^0 and there is at least one angle, say $\angle ABC$, which is greater than 60^0 it follows that $sin(\angle ABC) \geq \frac{\sqrt{3}}{2}$ so that

$$F_{\Delta ABC} = \frac{1}{2} AB \cdot BC \cdot sin(\angle ABC) \geq \frac{\sqrt{3}}{4} \cdot 25 > \frac{21}{2} \quad \text{(since } 625 > 588 > 12.49\text{)}.$$

Similarly one can prove that $F_{\Delta ACD} \geq \frac{21}{2}$, and thus $F_{ABCD} > 21$.

Problem 5.2. Two unit squares with parallel sides overlap by a rectangle of area $\frac{1}{8}$. Find the extreme values of the distance between the centers of the squares.

Solution. Let $MNPQ$ be the rectangle at the intersection of the unit squares with centers A and B (Figure 10). Set $MN = x, PQ = y$, hence

$$xy = \frac{1}{8},\ x,y \in (0,1].$$

Suppose that the parallel from A to MN intersects the parallel from B to NP at the point C. It is easy to observe that

$$AC = \frac{1}{2} + \left(\frac{1}{2} - x\right) = 1 - x,\ BC = 1 - y \text{ and}$$

$$AB^2 = (1-x)^2 + (1-y)^2 = x^2 + y^2 - 2(x+y) + 2 =$$

$$= x^2 + 2xy + y^2 - 2(x+y) - \frac{1}{4} + 2 = (x+y)^2 - 2(x+y) + \frac{7}{4} =$$

$$= (x+y-1)^2 + \frac{3}{4},$$

so $AB \geq \frac{\sqrt{3}}{2}$.

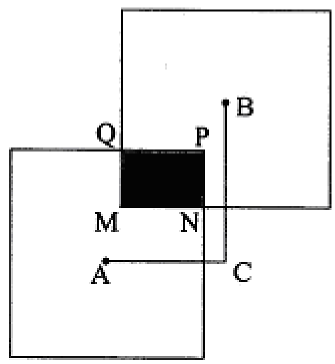


Figure 10

It follows that the minimal value of the distance between the centers A, B is equal to $\frac{\sqrt{3}}{2}$, and it is obtained for $x + y = 1,\ xy = \frac{1}{8}$, i.e.

$$x = \frac{2+\sqrt{2}}{4},\ y = \frac{2-\sqrt{2}}{4} \text{ or } x = \frac{2-\sqrt{2}}{4},\ y = \frac{2+\sqrt{2}}{4}.$$

To find the maximal value of AB one can observe that

$$0 \leq (1-x)(1-y) = 1 - x - y + xy = \frac{9}{8} - (x+y), \text{ i.e. } x + y \leq \frac{9}{8}.$$

On the other hand, we have

$$x + y \geq 2\sqrt{xy} = \frac{1}{\sqrt{2}}, \text{ therefore } 0 > \frac{1}{\sqrt{2}} - 1 \leq x + y - 1 \leq \frac{1}{8}.$$

As $\left(\frac{1}{8}\right)^2 \leq \left(\frac{1}{\sqrt{2}} - 1\right)^2$, then $\frac{1}{\sqrt{2}} - 1 \leq x + y - 1 \leq 1 - \frac{1}{\sqrt{2}}$ and we find that

$$AB^2 \leq \left(1 - \frac{1}{\sqrt{2}}\right)^2 + \frac{3}{4} = \frac{9}{4} - \sqrt{2} = \left(\sqrt{2} - \frac{1}{2}\right)^2.$$

Thus $AB \le \sqrt{2} - \frac{1}{2}$, with equality when $x = y = \frac{1}{2\sqrt{2}}$. Consequently,

$$\frac{\sqrt{3}}{2} \le AB \le \sqrt{2} - \frac{1}{2}.$$

Section 6. LOCI

Problem 6.1. If A and B are fixed points on a given circle and XY is a variable diameter of the same circle, determine the locus of the point of intersection of lines AX and BY. You may assume that AB is not a diameter.

Solution. Figure 11 and Figure 12 show two positions of the moving diameter XY. In either case, $\angle XAY = 90^0$ and also $\angle AYB = \frac{AB}{2} = \theta$ is constant. Thus, in Figure 11, the angle $\angle APB = \frac{\pi}{2} - \theta$, which is the complement of $\angle AYB$, is also constant. Similarly, in the Figure 12, the angle $\angle AP'Y$ is constant since it is the complement of $\angle AYB$. Since $\angle AP'B$ is the supplement of $\angle AP'Y$ it is also constant. In Figure 11, the locus of points P such that $\angle APB$ is constant is a circular arc on chord AB.

In Figure 12, the triangle with fixed base AB and constant $\angle AP'B$ has its vertex P' on a circular arc on chord AB. Since ΔAPY in Figure 11 is similar to $\Delta AP'Y$ in the Figure 12, it follows that angles APB and $AP'B$ are supplementary. Thus P and P' lie on the same circle. The radius of this circle, by the extended Law of Sine, is

$$r = \frac{AB}{2\,sin(\angle APB)} = \frac{AB}{2\,sin\left(\frac{\pi}{2} - \theta\right)} = \frac{AB}{2\,cos\,\theta}.$$

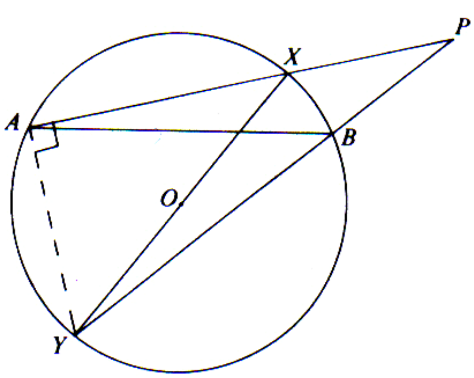


Figure 11

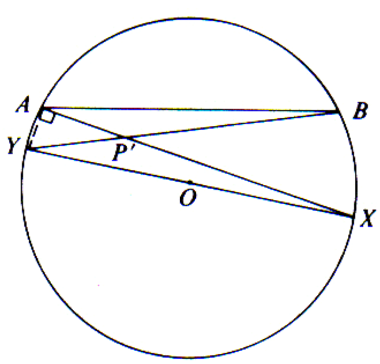


Figure 12

Problem 6.2. Suppose two unequal circles touch each other internally at a point A. A tangent to the smaller circle intersects the larger one at points B and C. Find the locus of the centers of the circles inscribed in triangle ABC.

Solution. Let R and r denote the radii of the given circles $(R \geq r)$ and let D be the point of tangency of the chord BC and the smaller circle (Figure 13). Let K and L be the points of intersection of the chords AC and AB with the smaller circle. Finally, let O be the center of the circle inscribed in the triangle ABC.

Since the angular measures of the arcs AK and AC are equal, $|AK| = rx, |AC| = Rx$, hence we get

$$|DC|^2 = |AC| \cdot |CK| = (R-r)Rx^2.$$

Similarly,

$$|AB| = Ry, \quad |AL| = ry, \quad |DB|^2 = (R-r)Ry^2.$$

Consequently, $\dfrac{|CD|}{|DB|} = \dfrac{x}{y} = \dfrac{|AC|}{|AB|}$, so that is the bisector AD of the angle BAC.

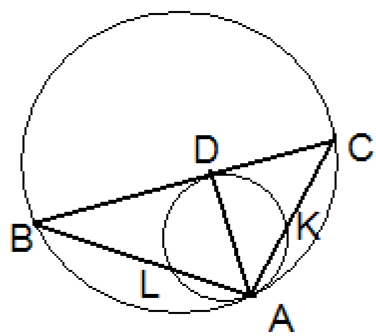


Figure 13

Further, we have

$$\frac{|AO|}{|OD|} = \frac{|AC|}{|CD|} = \frac{Rx}{\sqrt{(R-r)Rx^2}} = \sqrt{\frac{R}{R-r}}.$$

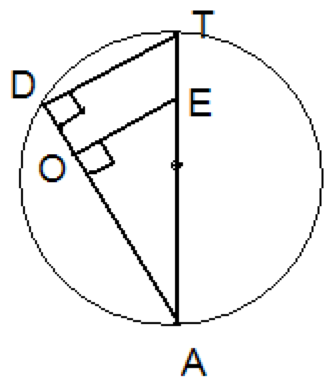


Figure 14

Thus, the desired locus is a circle of radius $\rho = \frac{1}{2}|AE|$ touching internally the two given circles at the same point *A*. From the obvious identity $\frac{|AE|}{|AT|} = \frac{|AO|}{|AD|}$ (Figure 14) it follows

$$\rho = r\frac{|AO|}{|AD|} = \frac{r\sqrt{R}}{\sqrt{R} + \sqrt{R-r}}.$$

***Problem 6.3**.* Suppose ABC is an isosceles triangle with $BC = a$ *and* $AB = AC = b$. Two variable points *M* and *N* are given by the conditions:

$$M \in (AC), N \in (AB) \text{ and } a^2 AM.AN = b^2 BN.CM.$$

If the straight lines BM and CN intersect in point P, find the locus of the variable point P.

First solution. Consider D on the line CB such that the point B is between D and C, and $AD = CD$ (Figure 15). From the similar triangles ABC and DAC we obtain $CD = \frac{b^2}{a}$. The common point Q of the lines AD and CN satisfies

$$\frac{QA}{QD} \cdot \frac{CD}{CB} \cdot \frac{NB}{NA} = 1,$$

therefore $\frac{QA}{QD} = \frac{a^2}{b^2} \cdot \frac{NA}{NB} = \frac{MC}{MA}$, i.e. $\frac{QA}{CD - QA} = \frac{MC}{b - MC}$ or $\frac{QA}{CD} = \frac{MC}{b}$.

Thus, $\frac{QA}{MC} = \frac{CD}{b} = \frac{b}{a} = \frac{AC}{BC}$ and further ΔBMC is similar to ΔCQA.

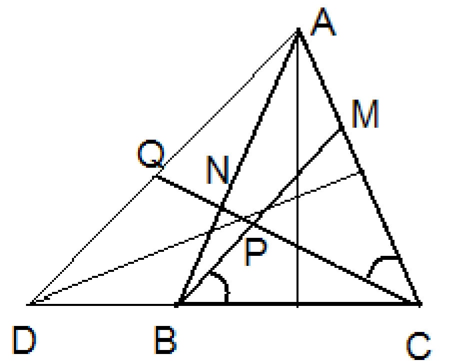


Figure 15

This proves that $\angle MBC = \angle NCA$, whence $\angle(BPC) = 180^0 - \angle(B)$ is constant, which shows that P describes the $arcBC$ of the circle that is tangent to AB and $AC,$ respectively, at B and C.

Second solution. Let R be the point of the segment AB such that $AR = AM$. It follows:

$$\frac{RA}{RB} \cdot \frac{AN}{NB} = \frac{AM}{MC} \cdot \frac{AN}{NB} = \frac{AC^2}{BC^2} \ ,$$

therefore $\angle ACR = \angle BCN$, by Steiner's Theorem**.** Whence $\angle ABM = \angle BCN$ and the solution finishes as above.

Section 7. CONSTRUCTION PROBLEMS

Problem 7.1. Construct a triangle ABC with given angle $C = \gamma$, where $0^0 < \gamma < 180^0$ altitude h_a and median m_c.

Solution. Analysis. Suppose that the triangle ABC is constructed with
$AD \perp BC,\ AD = h_a,\ CM = m_c,\ \angle ACB = \gamma$ (Figure 16, Figure17).

The right angled triangle ACD can be constructed by the given elements:

i) $\angle ACD = \gamma,\ AD = h_a$ (Figure 16); or

ii) $\angle ACD = 180^0 - \gamma,\ AD = h_a$ (Figure 17).

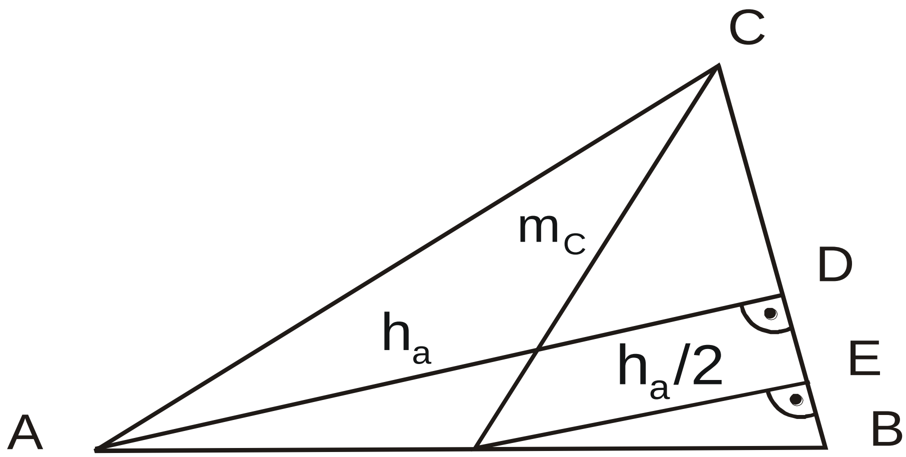


Figure 16

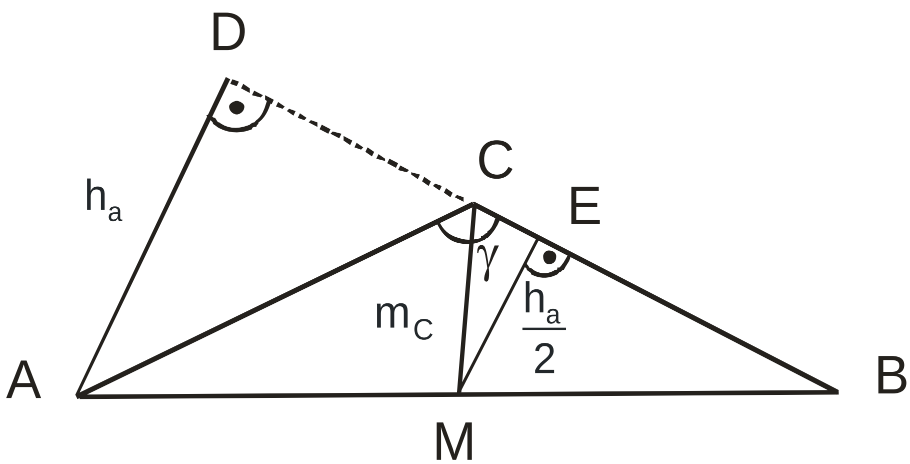


Figure 17

We construct the line segment $ME \perp BC$. Note that $ME = \frac{h_a}{2}$. Thus M is the intersection point of two sets, i.e. $M \in M_1 \cap M_2$, where M_1 is a circle with center C and radius m_c and M_2 is a line parallel to BC at a distance $\frac{h_a}{2}$.

Construction. We construct the triangles ADC with $AD = h_a$ and $\angle ACD = \gamma$ or $\angle ACD = 180^0 - \gamma$, where γ is an acute angle or γ is an obtuse angle, respectively. Then we construct the sets M_1, M_2 and determine their intersection point M, if there is one. The point B is intersection point for the lines AM and CD.

Research. The problem has 2, 1 or 0 solutions, depending on the following conditions, respectively:

$$m_c > \frac{h_a}{2},\ m_c = \frac{h_a}{2},\ m_c < \frac{h_a}{2} \quad \text{(see } \Delta CME\text{)}.$$

Proof. By construction we have $AD = h_a$, $\angle ACB = \angle ACD = \gamma$, if γ is an acute angle (Figure 16) and $AD = h_a$, $\angle ACD = 180^0 - \angle ACB = 180^0 - \gamma$, if γ is an obtuse angle (Figure17). The point M is the midpoint of the segment AB because, by construction, the distance from M to BC is equal to $\frac{h_a}{2} = \frac{AD}{2}$ and $ME = \frac{h_a}{2}$ for the triangle ABD. Moreover, by construction, $CM = m_a$. Consequently the triangle ABC is the required.

Problem 7.2. Construct a triangle with given median, bisector, and altitude through one of its vertices.

Short Analysis and Construction. If the given altitude, bisector and median are CD, CL, and CM, respectively, then we construct the right angled triangles CDL and CDM. The straight line p, $M \in p$, $p \perp MD$, intersects the straight line CL at a point N (the line p is the perpendicular bisector of AB). Then CN is a chord of the circle (with a center O and radius OC), that is circumscribed about the triangle ABC. The center O is the intersection point of the perpendicular bisector s of CD and the line MN. The constructed circle intersects MD at points A and B.

Section 8. TRANSFORMATIONS: ROTATION, SIMILARITY, INVERSION

Problem 8.1. Given a triangle ABC and the equilateral triangle PQR (Figure 18). Suppose that $\angle ADB = \angle BDC = \angle CDA = 120^0$ in the triangle ABC. Prove that $x = u + v + w$.

Solution. We shall construct the equilateral triangle PQR and show that its sides have length $u + v + w$. We rotate BCD through 60^0 counterclockwise about B to position BFE (Figure 19).

First note that BDE and BCF are equilateral, so that $DE = v$ and $CF = a$, Now $\angle ADE$ and $\angle DEF$ are straight angles, both $120^0 + 60^0$, so that $AF = u + v + w$.

Construct the equilateral triangle AFG with side AF, as it is shown in Figure 20. Now, $AF = FG$, $CF = BF$ and $\angle CFA = \angle BFG = 60^0 - \angle AFB$. Thus, CFA is similar to BFG. It follows that $BG = AC = b$, so that AFG is the required equilateral triangle with side lengths $x = u + v + w$.

At the end, we have to note that the point D is called ***an isogonic center*** of triangle ABC and can be constructed by drawing equilateral triangles BCP_1, CAP_2 and ABP_3 externally of the triangle ABC. Then the lines AP_1, BP_2 and CP_3 intersect at the point

D. The side x is given by the formula: $2x^2 = a^2 + b^2 + c^2 + 4\sqrt{3}F_{\Delta ABC}$, where $F_{\Delta ABC}$ is the area of the triangle ABC.

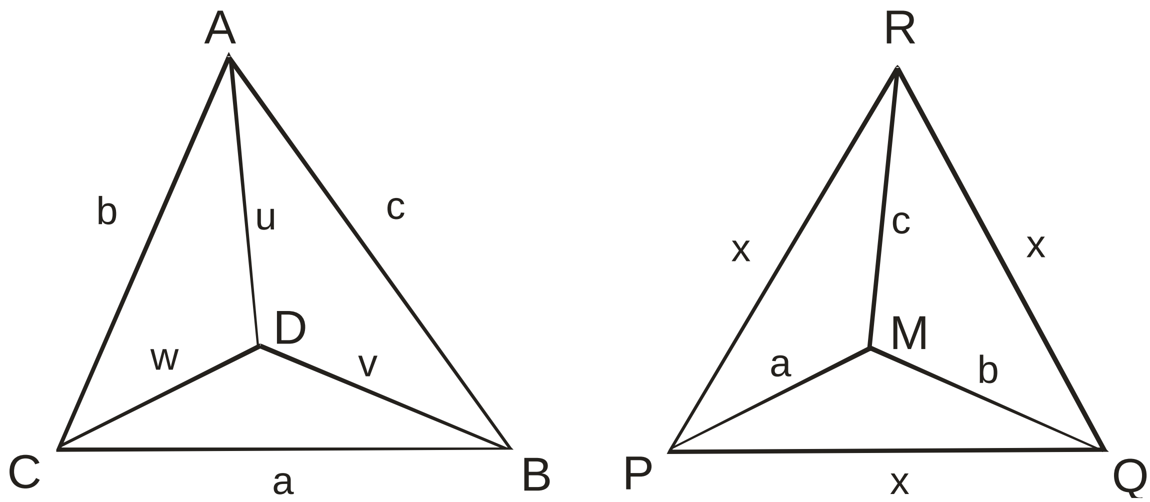


Figure 18

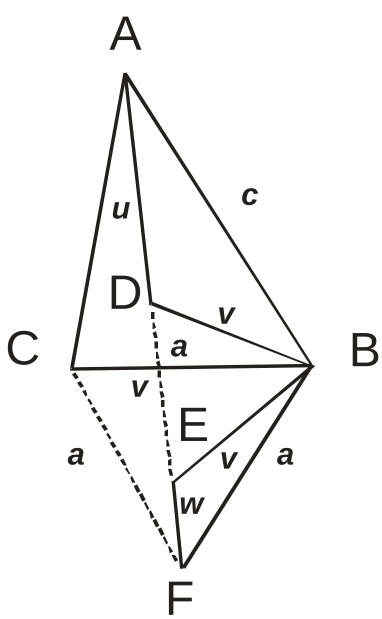


Figure 19

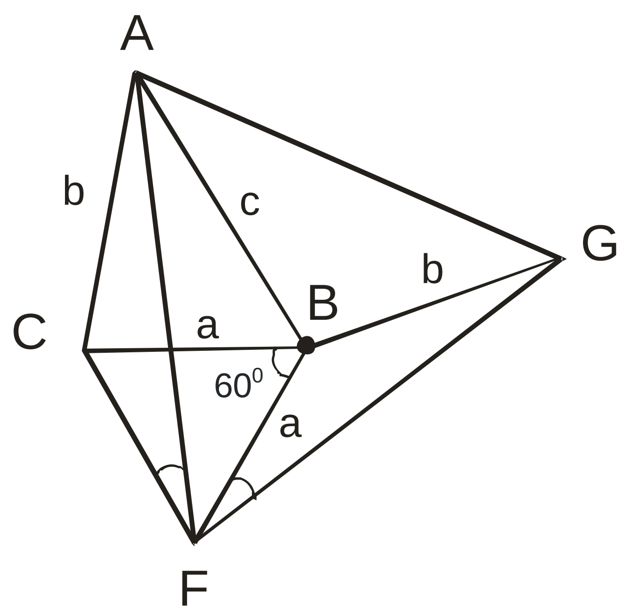


Figure 20

Problem 8.2. Give a geometrical interpretation of the system

$$\begin{vmatrix} x^2 + xy + y^2 = a^2 \\ y^2 + yz + z^2 = b^2 \\ z^2 + zx + x^2 = a^2 + b^2 \end{vmatrix}$$

and find $x + y + z$.

***Solution**.* If $x \geq 0,\ y \geq 0,\ z \geq 0$ then x, y, z are the distances to the vertices of a right triangle ABC in which the perpendiculars BC and CA are a and b, respectively, from a point M inside the triangle from which its three sides can be seen at an angle of 120^0.
To determine the sum $x+y+z$, rotate the triangle CMA about C through an angle of 60^0 in the direction external with respect to the triangle ABC. As a result, M and A go into M_1 and A_1, respectively (Figure 21). Then BMM_1A_1 is a straight line and, consequently,

$$x+y+z = |BM| + |CM| + |AM| = |BA_1| = \sqrt{a^2+b^2+ab\sqrt{3}}.$$

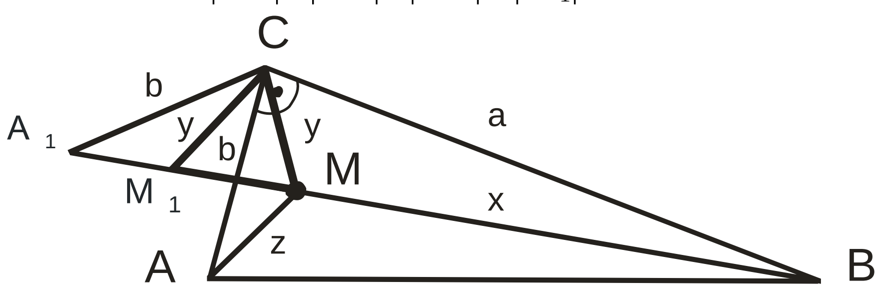


Figure 21

Similarly, we consider the case when one of the variables is negative - say $y<0$ and $\angle AMB = 120^0$, $\angle AMC = \angle BMC = 60^0$, $CM = -y > 0$ (Figure 22). The other cases are dealt with likewise.
The answer is: $x+y+z = \sqrt{a^2+b^2 \pm ab\sqrt{3}}$.

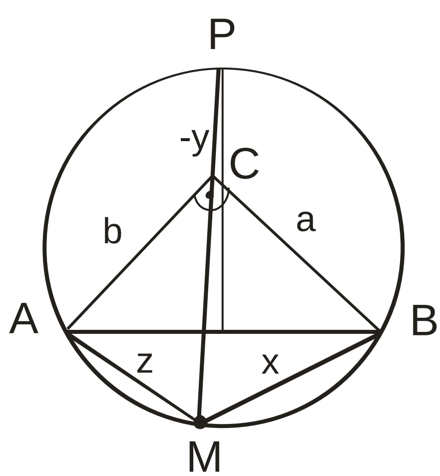


Figure 22

Problem 8.3. Let X, Y, Z, T be four points in the plane. The segments $[XY]$ and $[ZT]$ are said to be *connected*, if there is some point *O* in the plane such that the triangles OXY and OZT are isosceles with right angles at O. Let $ABCDEF$ be a convex hexagon such that the pairs of segments $AB,\ CE$ and $BD,\ EF$ are connected. Show that the points A, C, D and F are the vertices of a parallelogram and that the segments BC and AE are connected.

Solution. We shall consider geometrical transformations.
Let O_1 be the common vertex of the right isosceles triangles O_1AB and O_1CE and let O_2 be the common point of the triangles O_2BD and O_2EF. If R_i, $(i=1, 2)$, denotes the rotation with the center O_i and angle $\frac{\pi}{2}$, then $A=R_1(B),\ B=R_2(D)$ so that, in terms of composition of transformations, $A=(R_1\circ R_2)(D)$.
Similarly, $E=R_1^{-1}(C),\ F=R_2^{-1}(E)$, that is $F=\left(R_2^{-1}\circ R_1^{-1}\right)(C)=(R_1\circ R_2)^{-1}(C)$.
We remark that $(R_1\circ R_2)$ is a rotation by an angle π. This implies $(R_1\circ R_2)^{-1}=(R_1\circ R_2)$ and, as a by consequence, A and F are obtained from D and C, respectively, by the some rotation by an angle π. We conclude that $ACDF$ is a parallelogram.

Problem 8.4. Consider a quadrilateral $ABCD$ with sides $AB=a,BC=b,CD=c,DA=d$ and diagonals $AC=d_1,BD=d_2$. Prove that there exists a triangle with sides ac,bd,d_1d_2.

Hint. Use the inversion In with, for example, center the point D and coefficient $k=cdd_2$. Then the obtained triangle $A'B'C'$, where $A'=In(A),\ B'=In(B),\ C'=In(C)$ has sides $A'B'=ac,\ B'C'=bd,\ C'A'=d_1d_2$.

Section 9. SPECIAL THEOREMS

Problem 9.1 (Leibniz's theorem). Let M be an arbitrary point in the plane and G the center of mass of a triangle ABC. Prove the identity

$$3|MG|^2=|MA|^2+|MB|^2+|MC|^2-\frac{1}{3}\left(|AB|^2+|BC|^2+|CA|^2\right).$$

Solution. Let the points A,B,C and M have the following coordinates in the rectangular Cartesian system: $A(x_1,y_1),B(x_2,y_2),C(x_3,y_3),M(x,y)$, respectively. Then the point G has coordinates

$$G\left(\frac{x_1+x_2+x_3}{3},\ \frac{y_1+y_2+y_3}{3}\right).$$

Hence, the validity of the assertion follows from the identity

$$3\left(x-\frac{x_1+x_2+x_3}{3}\right)^2=$$

$$=\left(x-x_1\right)^2+\left(x-x_2\right)^2+\left(x-x_3\right)^2-\frac{1}{3}\left[\left(x_1-x_2\right)^2+\left(x_2-x_3\right)^2+\left(x_3-x_1\right)^2\right]$$

and analogous relationship for the ordinates.

Problem 9.2 (Bretschnaider's theorem). Let a,b,c,d be the sides of a quadrilateral, *m* and *n* its diagonals and A, C its two opposite angles. The following relationship is fulfilled

$$m^2n^2=a^2c^2+b^2d^2-2abcd.cos\left(A+C\right).$$

Solution. In the quadrilateral $ABCD$ (Figure 23) we have

$$\left|AB\right|=a,\ \left|BC\right|=b,\ \left|CD\right|=c,\ \left|DA\right|=d,\ \left|AC\right|=m,\ \left|BD\right|=n.$$

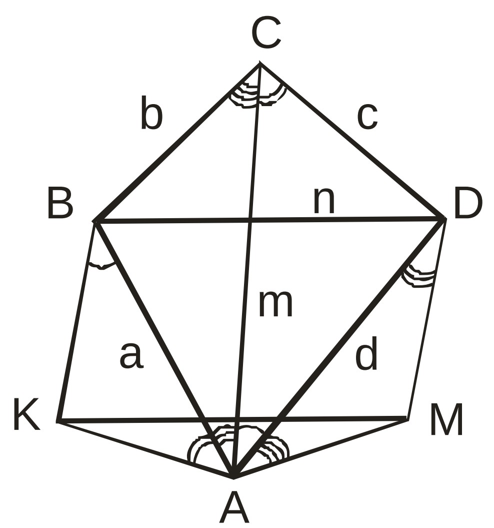


Figure 23

Construct externally on the side AB, a triangle AKB similar to the triangle ACD, where

$$\angle BAK=\angle DCA,\ \angle ABK=\angle CAD,$$

and on the side AD we construct the triangle AMD similar to the triangle ABC, where

$$\angle DAM=\angle BCA,\ \angle ADM=\angle CAB.$$

From the corresponding similarity we get

$$\left|AK\right|=\frac{ac}{m},\ \left|AM\right|=\frac{bd}{m},\ \left|KB\right|=\left|DM\right|=\frac{ad}{m}\ .$$

In addition

$$\angle KBD+\angle MBD=\angle CAD+\angle ABD+\angle BDA+\angle CAB=180^0,$$

that is, the quadrilateral $KBDM$ is a parallelogram.

Hence, $\left|KM\right|=\left|BD\right|=n$. But $\angle KAM=\angle A+\angle C$.

By the Law of Cosines for the triangle KAM, we have

$$n^2=\left(\frac{ac}{m}\right)^2+\left(\frac{bd}{m}\right)^2-2\left(\frac{ac}{m}\right)\left(\frac{bd}{m}\right)^2\cdot cos\left(A+C\right),$$

and, hence,

$$m^2n^2 = a^2c^2 + b^2d^2 - 2abcd.cos\left(A+C\right).$$

Section 10. METRIC PROBLEMS

Problem 10.1. Let ABC be an equilateral triangle with side a, and M - some point in the plane at a distance d from the center of the triangle ABC. Prove that the area of the triangle whose sides are equal to the line segments MA, MB and MC can be expressed by the formula

$$F = \frac{\sqrt{3}}{12}\left|a^2 - 3d^2\right|.$$

Solution. Consider the case when the point M (Figure 24) lies inside of the triangle ABC.

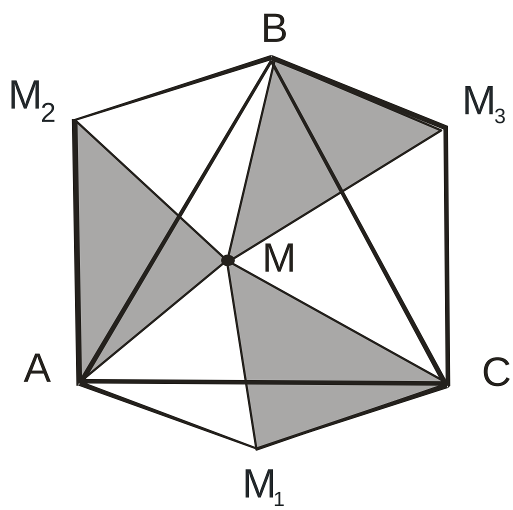


Figure 24

Rotate the triangle ABM about *A* through an angle of 60^0 to bring B onto C. We get the triangle AM_1C which is congruent to the triangle ABM; the triangle AMM_1 is equilateral, consequently, the sides of the triangle CMM_1 are equal to the line segments MA, MB, MC. The points M_2, M_3 are obtained in a similar way.

The area of the hexagon $AM_1CM_3BM_2$ is twice the area of the triangle ABC, that is, equal to $a^2\frac{\sqrt{3}}{2}$.

On the other hand, the area of this hexagon is expressed as the sum of the areas of three equilateral triangles AMM_1, CMM_3, BMM_2 and the three triangles congruent to the desired one.

Consequently,

$$3F + \left(|MA|^2 + |MB|^2 + |MC|^2\right) . \frac{\sqrt{3}}{4} = a^2 \frac{\sqrt{3}}{2} .$$

Using the result of ***Leibniz's theorem***, we get

$$3F + \left(3d^2 + a^2\right)\frac{\sqrt{3}}{4} = a^2 \frac{\sqrt{3}}{2}, \text{ whence } F = \frac{\sqrt{3}}{12}\left(a^2 - 3d^2\right).$$

Other cases of the position of the point *M* can be considered in a similar way.

Problem 10.2 (Fermat's theorem). Suppose $ABCD$ is a rectangle with

$$|AB| = 2a,\ |BC| = a\sqrt{2} .$$

Construct a semicircle externally on the side AB as diameter. Let M be an arbitrary point on the semicircle. If the line MD intersects AB at N, and the line MC at L, find $|AL|^2 + |BN|^2$.

Solution. Let P be the projection of M on AB and let $|AP| = a + x$ (Figure 25).

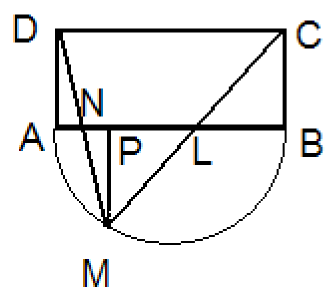


Figure 25

Then

$$|PB| = a - x,\ |MP| = y = \sqrt{a^2 - x^2},\ |AN| = (a + x)\cdot\frac{a\sqrt{2}}{a\sqrt{2} + y},$$

$$|NB| = 2a - (a + x)\cdot\frac{a\sqrt{2}}{a\sqrt{2} + y} = \frac{a\sqrt{2}\left(a - x + y\sqrt{2}\right)}{a\sqrt{2} + y},$$

and, similarly

$$|AL| = \frac{a\sqrt{2}\left(a + x + y\sqrt{2}\right)}{a\sqrt{2} + y} .$$

Hence,

$$|AL|^2+|NB|^2=\frac{4a^2}{\left(a\sqrt{2}+y\right)^2}\left(a^2+2\sqrt{2}ay+2y^2+x^2\right)=$$

$$=\frac{4a^2}{\left(a\sqrt{2}+y\right)^2}\left[a^2+2\sqrt{2}ay+2y^2+\left(a^2-y^2\right)\right]=4a^2.$$

Section 11. GEOMETRIC INEQUALITIES

Problem 11.1. Let AB be a diameter of a circle of radius 1 and let the points C and E be distinct points on the circle and on the same side of AB. If parallel chords CD and EF cut AB at an angle of 45^0, at points P and Q respectively, prove that

$$PC\cdot QE+PD\cdot QF<2.$$

Solution. By the **Arithmetic Mean – Geometric Mean Inequality,** we have (Figure 26):

$$PC\cdot QE+PD\cdot QF\le\frac{1}{2}\left(PC^2+QE^2\right)+\frac{1}{2}\left(PD^2+QF^2\right),$$

Let M be the midpoint of CD. Then OM is perpendicular to CD. We have

$$PC^2+PD^2=\left(CM-PM\right)^2+\left(DM+PM\right)^2=2\left(CM^2+OM^2\right)=2.$$

Similarly, $QE^2+QF^2=2$, so that $PC.QE+PD.QF\le 2$. Since $PC\ne QE$, equality cannot hold, and we have $PC.QE+PD.QF<2$.

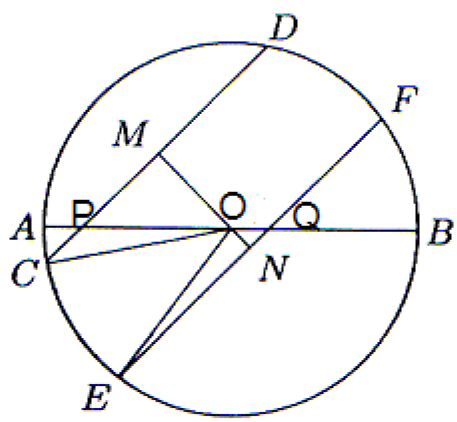


Figure 26

Problem 11.2. Consider a triangle ABC and let O be a point in its interior. Suppose straight lines OA, OB, OC meet the sides of the triangle at $A_1,\ B_1, C_1$, respectively, and suppose $R_1,\ R_2,\ R_3,\ R$ are the radii of the circumcircles of the triangles $OBC, OCA, OAB,\ ABC$, respectively. Prove that

$$\frac{OA_1}{AA_1}R_1+\frac{OB_1}{BB_1}R_2+\frac{OC_1}{CC_1}R_3\geq R.$$

Solution. It is obvious that

$$\frac{OA_1}{AA_1}=\frac{F_{\Delta OBC}}{F_{\Delta ABC}}=\frac{OB\cdot OC\cdot BC}{4R_1}\cdot\frac{4R}{AB\cdot BC\cdot CA}\text{ , i.e.}$$

$$\frac{OA_1}{AA_1}R_1=OB\cdot OC\cdot BC\cdot\frac{R}{AB\cdot BC\cdot CA}\text{ , etc.}$$

So we have to prove that

$$\sum OB\cdot OC\cdot BC\geq AB\cdot BC\cdot CA\ .$$

Let us consider the complex numbers $O(o),\ A(a),\ B(b),\ C(c)$. Then the last inequality becomes:

$$\sum|b|\cdot|c|\cdot|b-c|\geq|a-b|\cdot|b-c|\cdot|c-a|\ ,$$

that is

$$\sum|b^2c-c^2b|\geq|ab^2+bc^2+ca^2-a^2b-b^2c-c^2a|,$$

which is obvious.

Problem 11.3. Prove the ***Finsler-Hadwiger*** inequality

(10) $$a^2+b^2+c^2\geq 4F\sqrt{3}+(a-b)^2+(b-c)^2+(c-a)^2,$$

where *a*, *b*, *c*, *F* are the sides and area of any triangle, respectively.

Solution. Denote

$$s-a=x,\ s-b=y,\ s-c=z\ ,$$

where s is the semi perimeter of the given triangle and $x,y,z>0$.

Leaving $4F\sqrt{3}$ at the right-hand side of the inequality (10) and, after transforming the left-hand side (for instance: $a^2-(b-c)^2=4(s-b)(s-c)=4yz$) and replacing F by ***Heron's formula***

$$F=\sqrt{s(s-a)(s-b)(s-c)},$$

we get the equivalent inequality

(11) $$xy+yz+zx\geq\sqrt{3(x+y+z)xyz}\ .$$

Dividing both sides of the inequality (11) by $\sqrt{xyz}$ and making the substitutions

$$u=\sqrt{\frac{xy}{z}}\ ,\ v=\sqrt{\frac{yz}{x}}\ ,\ w=\sqrt{\frac{zx}{y}}\quad(x=uw,\ y=vu,\ z=wv),$$

we get another equivalent inequality, namely

$$u+v+w\geq\sqrt{3(uv+vw+wu)}\ .$$

This, on squaring, is reduced to the known inequality

$$u^2+v^2+w^2\geq uv+vw+wu\ ,$$

Which is equivalent to the obvious $\sum(u-v)^2 > 0$.

Problem 11.4. Prove that in any triangle with medians m_a, m_b, m_c and area F the following inequality holds

$$\frac{1}{m_b m_c}+\frac{1}{m_c m_a}+\frac{1}{m_a m_b} \le \frac{\sqrt{3}}{F}. \tag{12}$$

Solution. Let $a, b, c, F, m_a, m_b, m_c$ be the usual elements of an arbitrary triangle ABC. With the medians of a triangle ABC one can form a "median-dual triangle" AA_1M (Figure 27), where the figures BB_1MA_1 and $AMCC_1$ are parallelograms:

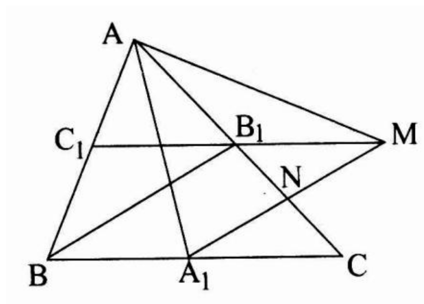


Figure 27

Furthermore, it is obvious that the area F_m of the median-dual triangle AA_1M is $F_m = \frac{3}{4}F$. Indeed,

$$\frac{F_m}{F} = \frac{2F_{\Delta AA_1N}}{2F_{\Delta AA_1C}} = \frac{AN}{AC} = \frac{3}{4}$$

Thus, the inequality (12) becomes

$$\frac{1}{m_b m_c}+\frac{1}{m_c m_a}+\frac{1}{m_a m_b} \le \frac{3\sqrt{3}}{4F_m}. \tag{13}$$

Writing a,b,c,F instead of m_a, m_b, m_c, F_m, equality (13) reduces to

$$\frac{F}{ab}+\frac{F}{bc}+\frac{F}{ca} \le \frac{3\sqrt{3}}{4}$$

or

$$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}, \tag{14}$$

which is a known inequality. For a short proof, notice that the sine function is concave down on $[o,\pi]$ and so, by Jensen's inequality,

$$\frac{\sin A + \sin B + \sin C}{3} \le \sin\frac{A+B+C}{3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad .$$

Now, from (14) is easy to go back to (13), or (12).

Problem 11.5. Suppose a triangle has sides a,b,c. Determine in terms of a,b,c the area F_1 of the largest equilateral triangle circumscribed about the given triangle, and the area F_2 of the least equilateral triangle inscribed in it.

Hint. There are two families of equilateral triangles circumscribed about the given triangle.

Section 12. APPLICATIONS OF VECTORS IN GEOMETRY

Problem 12.1. Let M be an arbitrary point on the side BC of the triangle ABC. Let C_0,C_1,C_2 be the incircles of the triangles ABC, ABM, ACM, respectively and I be the center of C_0.

1. Prove that C_1,C_2 are tangent if and only if $M \in C_0$.
2. Suppose $M \in C_0$ and let D and C be the midpoints of the segment BC and AM, respectively. Prove that the points I, S, D are collinear and $\dfrac{IS}{ID} = \dfrac{s-a}{a}$.

Solution.

i) We will use the following result: Suppose that the inscribed circle of a triangle ABC touches the sides *AB*, *BC*, *CA* at points C_1, A_1, B_1, respectively (Figure 28). Then

$$AB_1 = AC_1 = \frac{AB + AC - BC}{2} = s - a.$$

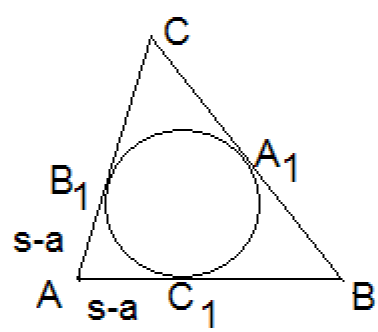


Figure 28

Let the line AM touch the circles C_1,C_2 at points T_1,T_2, respectively (Figure 29). The circles C_1,C_2 are tangent if and only if $T_1 = T_2$, that is when $AT_1 = AT_2$.
Using the above results, this is equivalent to

$$\frac{AM + AB - BM}{2} = \frac{AM + AC - CM}{2},$$

$$AB - BM = AC - (BC - BM)$$

or

$$BM = \frac{AB + BC - AC}{2} = BA_1$$

and so $M \equiv A_1 \in C_0$, as required.

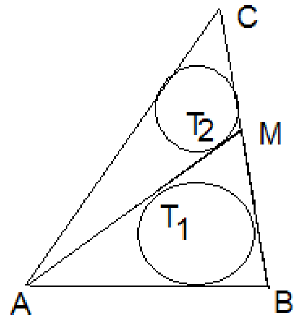


Figure 29

ii) Let a, b, c be the side lengths of the triangle ABC and let L be the intersection of BC with the internal bisector AI of $\angle BAC$ (Figure 30).
By angle bisector theorem it follows that

$$\frac{LB}{LC} = \frac{AB}{AC}, \text{ i.e. } \frac{LB}{BC} = \frac{AB}{AB + AC}.$$

The key idea is to represent vectors $\overrightarrow{AD}, \overrightarrow{AS}, \overrightarrow{AI}$ as linear combinations of the vectors $\overrightarrow{AB}, \overrightarrow{AC}$. Thus,

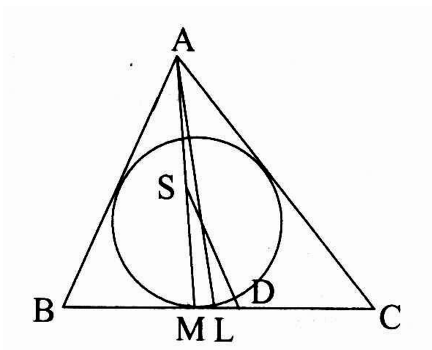


Figure 30

(15) $$\overrightarrow{AD} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right),$$

(16) $$\overrightarrow{AS} = \frac{1}{2}\left(\overrightarrow{AM}\right) = \frac{1}{2}\left(\frac{CM}{BC}\overrightarrow{AB} + \frac{BM}{BC}\overrightarrow{AC}\right) = \frac{(s-c)\overrightarrow{AB} + (s-b)\overrightarrow{AC}}{2a}$$

and

$$\overrightarrow{AL} = \frac{LC}{BC}\overrightarrow{AB} + \frac{LB}{BC}\overrightarrow{AC} = \frac{AC}{AB + AC}\overrightarrow{AB} + \frac{AB}{AB + AC}\overrightarrow{AC} = \frac{b\overrightarrow{AB} + c\overrightarrow{AC}}{b + c}.$$

As BI is the bisector of $\angle ABL$, it follows that

$$\frac{AI}{IL} = \frac{AB}{BL} = \frac{c}{\frac{ac}{b+c}} = \frac{b+c}{a}, \text{ i.e. } \frac{AI}{AL} = \frac{b+c}{2s}.$$

Consequently,

(17) $$\overrightarrow{AI} = \frac{b\overrightarrow{AB} + c\overrightarrow{AC}}{2s}.$$

From (14), (15) and (16) easily follows that $s\overrightarrow{AI} = a\overrightarrow{AS} + (s-a)\overrightarrow{AD}$. The equation $\frac{a}{s} + \frac{s-a}{s} = 1$ shows that the point *I* lies on the line segment SD and is such that

$$\frac{IS}{ID} = \frac{s-a}{a}.$$

Problem 12.2. Let $ABCD$ be an inscribed quadrilateral and let M be a point on its circumcircle. Let H_1, H_2, H_3, H_4 be the orthocenters of triangles MAB, MBC, MCD, MDA, respectively. Prove that

i) the quadrilateral $H_1H_2H_3H_4$ is a parallelogram;

ii) $H_1H_3 = 2EF$, where E and F are the midpoints of the segments AB and CD, respectively.

***Solution*.**

1. We shall use vector algebra. The circum center of all triangles MAB, MBC, MCD, MDA is the point O. Hence, by ***Sylvester's formula***, we have

$$\overrightarrow{OH}_1 = \overrightarrow{OM} + \overrightarrow{OA} + \overrightarrow{OB}; \ \overrightarrow{OH}_2 = \overrightarrow{OM} + \overrightarrow{OB} + \overrightarrow{OC};$$

$$\overrightarrow{OH}_3 = \overrightarrow{OM} + \overrightarrow{OC} + \overrightarrow{OD}; \ \overrightarrow{OH}_4 = \overrightarrow{OM} + \overrightarrow{OD} + \overrightarrow{OA}.$$

Further computations give

$$\overrightarrow{H_1H_2} = \overrightarrow{OH_2} - \overrightarrow{OH_1} = \overrightarrow{OC} - \overrightarrow{OA} = \overrightarrow{OH_3} - \overrightarrow{OH_4} = \overrightarrow{H_4H_3}.$$

Thus the figure $H_1H_2H_3H_4$ is a parallelogram.

2. Using vectors again, we have (Figure 31):

$$\overrightarrow{H_1H_3} = \overrightarrow{OH_3} - \overrightarrow{OH_1} = \overrightarrow{OC} + \overrightarrow{OD} - \overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{AD} + \overrightarrow{BC} = 2\overrightarrow{EF}.$$

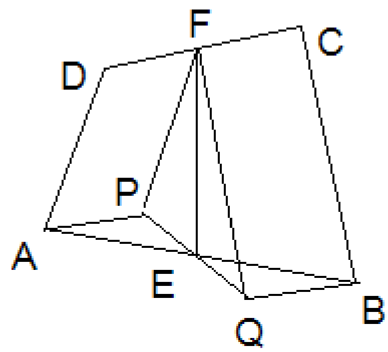


Figure 31

Hence $H_1H_3 = 2EF$.

MATHEMATICAL INDUCTION

Michael Lambrou
UNIVERSITY OF CRETE

Section 1. Historical Introduction

In philosophy and in the applied sciences the term *induction* is used to describe the process of drawing general conclusions from particular cases. For Mathematics, on the other hand, such conclusions should only be drawn with caution, because mathematics is a demonstrative science and any statement must be accompanied by a rigorous proof. For example John Wallis (1616-1703) was criticized strongly by his contemporaries because in his *Arithmetica Infinitorum* (1656), after inspecting the six relations,

$$\frac{0+1}{1+1}=\frac{1}{3}+\frac{1}{6}\ , \qquad \frac{0+1+4}{4+4+4}=\frac{1}{3}+\frac{1}{12},$$

$$\frac{0+1+4+9}{9+9+9+9}=\frac{1}{3}+\frac{1}{18}\ , \qquad \frac{0+1+4+9+16}{16+16+16+16+16}=\frac{1}{3}+\frac{1}{24},$$

$$\frac{0+1+4+9+16+25}{25+25+25+25+25+25}=\frac{1}{3}+\frac{1}{30}, \qquad \frac{0+1+4+9+16+25+36}{36+36+36+36+36+36+36}=\frac{1}{3}+\frac{1}{36}$$

stated without any further justification that the general rule, namely,

$$\frac{0^2+1^2+2^2+...+n^2}{n^2+n^2+n^2...+n^2}=\frac{1}{3}+\frac{1}{6n},$$

follows "*per modum inductionis*".

Although Wallis' claim is correct, amounting to the familiar statement (known to Archimedes) that

$$1^2+2^2+...+n^2=\left(\frac{1}{3}+\frac{1}{6n}\right)n^2(n+1)=\frac{1}{6}n(n+1)(2n+1),$$

it nevertheless needed proof.

One way to deal with this problem is with the so-called method of *complete* or *mathematical induction.* This topic, sometimes called just *induction*, is the subject discussed below.

Induction is a simple yet versatile and powerful procedure for proving statements about integers. It has been used effectively as a demonstrative tool in almost the entire spectrum of mathematics: for example in as diverse fields as algebra, geometry, trigonometry, analysis, combinatorics, graph theory and many others.

The principle of induction has a long history in mathematics. For a start, although the principle itself is not explicitly stated in any ancient Greek text, there are several places that contain precursors of it. Indeed, some historians see the following passage from

Plato's (427-347 BC) dialogue *Parmenides* (§147a7-c3) as the earliest use of an inductive argument:

Then they must be two, at least, if there is to be contact. - They must. - And if to the two terms a third be added in immediate succession, they will be three, while the contacts [will be] two. - Yes. - And thus, one [term] being continually added, one contact also is added, and it follows that the contacts are one less than the number of terms. For the whole successive number [of terms] exceeds the number of all the contacts as much as the first two exceed the contacts, for being greater in number than the contacts: for afterwards, when an additional term is added, also one contact to the contacts [is added]. - Right. - Then whatever the number of terms, the contacts are always one less. -True.

The previous passage is from a philosophical text. There are, however, several ancient mathematical texts that also contain quasi-inductive arguments. For instance Euclid (~330 - ~ 265 BC) in his *Elements* employs one to show that every integer is a product of primes.

An argument closer to the modern version of induction is in Pappus' (~290-~350 AD) *Collectio*. There the following geometric theorem is proved.

Let AB be a segment and C a point on it. Consider on the same side of AB three semi-circles with diameters AB, AC and CB, respectively. Now construct circles C_n as follows: C_1 touches the three semi-circles; C_{n+1} touches C_n and the semicircles on AB and AC. If d_n denotes the diameter of C_n and h_n the distance of its centre from AB, then $h_n = nd_n$.

The way Pappus proves the theorem is to show geometrically the recurrence relation $h_{n+1} /d_{n+1} = (h_n + d_n)/d_n$. Next, he invokes a result of Archimedes (287 - 212 BC) from his *Book of Lemma's* (Proposition 6) which states that the conclusion of the theorem above is true for the case n = 1. Coupling this with the recurrence relation, he is able to conclude the case for the general n.

After the decline of Greek mathematics, the Muses flew to the Islamic world. Although induction is not explicitly stated in the works of mathematicians in the Arab world, there are authors who reasoned using a preliminary form of it. For example al - Karaji (953-1029) in his *al-Fakhri* states, among others, the binomial theorem and describes the so called Pascal triangle after observing a pattern from a few initial cases (usually 5). He also knew the formula $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$. About a century later we find similar traces of induction in al-Samawal's (~1130-~1180) book *al-Bahir*, where the identity $1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n+1)/6$ appears. Subsequently Levi Ben Gershon (1288-1344), who lived in France, uses quasi-inductive arguments in his book *Maasei Hoshev* written in Hebrew.

The first explicit inductive argument in a source written in a western language is in the book *Arithmeticorum Libri Duo* (1575) of Francesco Maurolyco (1495–1575), a mathematician of Greek origin who lived in Syracuse. For instance it is shown inductively in this text that the sum of the first n odd integers is equal to the n^{th} square number. In symbols, $1 + 3 + 5 + … + (2n – 1) = n^2$, a fact already known to the ancient Pythagoreans.

Another early reference to induction is in the *Traité du Triangle Arithmetique* of Blaise Pascal (1623–1662), where the pattern known to-day as 'Pascal's Triangle' is discussed. There the author shows that the binomial coefficients ${}^nC_k$ satisfy ${}^nC_k : {}^nC_{k+1} = (k + 1) : (n – k)$, for all n and k with $0 \leq k < n$. Here the passage from n to n + 1 uses ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$.

All the above authors used an intuitive idea about the concept of natural number. This is sufficient for our purposes here, and below we shall follow suit. A characteristic of modern mathematics, however, especially from the late 19th century, was to develop the theory axiomatically. In particular, this was accomplished for the natural numbers by Giuseppe Peano (1858-1932) who published the so called 'Peano's axioms' in 1889, in a pamphlet entitled *Arithmetices principia, nova methodo exposita*. The exact procedure need not concern us here. We only mention that one of the axioms was so designed as to incorporate induction as a method of proof. In other words, the intuitive way to deal with induction below is actually a legitimate technique.

In what follows, the theory is presented in short sections, each with its own problems. These are rather easy especially at the beginning, but those in the last paragraph are more challenging. *Several questions can be solved by other means, but the idea is to use induction in all of them.*

Section 2. Basics

The principle of mathematical induction is a method of proving statements concerning integers. For example consider the statement "$1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n + 1)/6$", which we denote by P(n). One can easily verify this for various n, for instance $1^2 = 1 = 1.(1+ 1)(2.1 + 1)/6$, $1^2 + 2^2 = 5 = 2.(2 + 1)(2.2 + 1)/6$, $1^2 + 2^2 + 3^2 = 14 = 3.(3 + 1)(2.3 + 1)/6$ and so forth. Here we verified the statement for the cases n = 1, n = 2 and n = 3 (in a while we shall see that the last two can be dispensed with) but assume that we have verified it up to the particular value n = k. The last statement means that we are certain that for this particular value k we have "$1^2 + 2^2 + 3^2 + ... + k^2 = k(k + 1)(2k + 1)/6$". But is the formula true for the case of the next integer n = k + 1? We claim that it is. To see this, *making use of the fact that we have* $1^2 + 2^2 + 3^2 + ... + k^2 = k(k + 1)(2k + 1)/6$, we argue

$$\begin{aligned} 1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 &= k(k + 1)(2k + 1)/6 + (k + 1)^2 \qquad \text{(by assumption)} \\ &= (k + 1)[k(2k + 1) + 6(k + 1)]/6 \\ &= (k + 1)(k + 2)(2k + 3)/6, \end{aligned}$$

and this last is precisely the original claim for n = k + 1.

Let us recapitulate: We wanted to prove that the statement P(n) is true for all integers $n \geq 1$. We first verified it for n = 1; then, *assuming* that it is true for n = k, we verified it for n = k + 1. In other words, reiterating our result, the validity of P(1) implies that of P(2); the validity of P(2) implies that of P(3); the validity of P(3) implies that of P(4), *and so on* for all integers $n \geq 1$.

The schema we use in the proof can be summarised symbolically as

$$\begin{array}{l} P(1) \\ P(k) \Rightarrow P(k+1) \\ \hline \Rightarrow\ P(n) \text{ true for all } n \in \mathbf{N} \end{array}$$

The step "$P(k) \Rightarrow P(k+1)$" in the proof is called the *inductive step*; the assumption that P(k) is true, is called the *inductive hypothesis*.

Here is another example.

Example 2.1 (Bernoulli's inequality). Show that if a is a real number with $a > -1$, then $(1 + a)^n \geq 1 + na$ for all $n \in \mathbf{N}$.

Solution. For $n = 1$ it is a triviality (in fact we get an equality). Assume now validity of the inequality for $n = k$; that is, assume $(1 + a)^k \geq 1 + ka$. This is our inductive hypothesis, and we are to show $(1 + a)^{k+1} \geq 1 + (k + 1)a$. We have

$$\begin{aligned}(1 + a)^{k+1} &= (1 + a)(1 + a)^k \\ &\geq (1 + a)(1 + ka) &&\text{(by the inductive hypothesis)} \\ &= 1 + (k + 1)a + ka^2 \\ &\geq 1 + (k + 1)a &&\text{(since } ka^2 \geq 0\text{).}\end{aligned}$$

This, by the principle of induction, completes the proof. □

As a final remark, the above examples start from $n = 1$. This need not be always the case and there are cases (see problems) that induction may start at any another integer. The situation is self explanatory and there is no need to qualify it any further.

The next problems require the verification of a variety of formulae. None of these should present the reader with any difficulty and the problems are there only to familiarise him/her with the idea of induction. In fact, the reader should try to do several of these problems mentally.

Problem 2.1.(Routine). Show inductively that each of the following formulae is valid for all positive integers n.

1. $1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2(n + 1)^2/4$,
2. $1^4 + 2^4 + 3^4 + \ldots + n^4 = n(n + 1)(2n + 1)(3n^2 + 3n - 1)/30$,
3. $1^5 + 2^5 + 3^5 + \ldots + n^5 = n^2(n + 1)^2(2n^2 + 2n - 1)/12$,
4. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$,
5. $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{2\cdot 3\cdot 4} + \ldots + \dfrac{1}{n(n+1)(n+2)} = \dfrac{n(n+3)}{4(n+1)(n+2)}$,
6. $\dfrac{3}{1^2 2^2} + \dfrac{5}{2^2 3^2} + \dfrac{7}{3^2 4^2} + \ldots + \dfrac{2n+1}{n^2(n+1)^2} = \dfrac{n(n+2)}{(n+1)^2}$,
7. $(n + 1)(n + 2)\ldots(2n - 1)(2n) = 2^n.1.3.5....(2n - 1)$,
8. $\displaystyle\sum_{k=1}^{n} \frac{(2k)!}{k!2^k} = \sum_{k=1}^{n} 1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)$,
9. $1 - \dfrac{x}{1!} + \dfrac{x(x-1)}{2!} - \ldots + (-1)^n \dfrac{x(x-1)\ldots(x-n+1)}{n!} = (-1)^n \dfrac{(x-1)(x-2)\ldots(x-n)}{n!}$,
10. $(\cos x)(\cos 2x)(\cos 4x)(\cos 8x)\ldots(\cos 2^{n-1}x) = \dfrac{\sin 2^n x}{2^n \sin x}$ (for $x \in \mathbf{R}$ with $\sin x \neq 0$),
11. $\displaystyle\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin 2nx}{2\sin x}$, (for $x \in \mathbf{R}$ with $\sin x \neq 0$),

12. $\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}}_{\text{n radicals}} = 2\cos\frac{\pi}{2^{n+1}}$,

13. $(1^5 + 2^5 + 3^5 + \ldots + n^5) + (1^7 + 2^7 + 3^7 + \ldots + n^7) = 2(1 + 2 + 3 + \ldots + n)^4$,

14. $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}$.

Problem 2.2. If a sequence (a_n) satisfies

a) $a_{n+1} = 2a_n + 1$ ($n\in\mathbf{N}$), show that $a_n + 1 = 2^{n-1}(a_1 + 1)$.

b) $a_1 = 0$ and $a_{n+1} = (1 - x)a_n + nx$ ($n\in\mathbf{N}$), where $x \neq 0$, show that

$$a_{n+1} = [nx - 1 + (1 - x)^n]/x.$$

Problem 2.3. Let (a_n) be a given sequence. Define new sequences (x_n), (y_n) by $x_1 = 1$, $x_2 = a_1$, $y_1 = 0$, $y_2 = 1$ and, for $n \geq 3$, $x_n = a_n x_{n-1} + x_{n-2}$, $y_n = a_n y_{n-1} + y_{n-2}$. Show that

$x_{n+1}y_n - x_ny_{n+1} = (-1)^n$.

Problem 2.4. If each of $a_1, a_2, \ldots , a_n$, is a sum of two perfect squares, show that the same is true for their product.

Problem 2.5. Show that $2n^5/5 + n^4/2 - 2n^3/3 - 7n/30$ is an integer for all $n\in\mathbf{N}$.

Problem 2.6. Show that if $x \neq y$, then the polynomial $x - y$ divides $x^n - y^n$.

Problem 2.7. Show that a convex n-gon has $\frac{1}{2} n(n - 3)$ diagonals ($n \geq 3$).

Problem 2.8. Prove the binomial theorem inductively. Namely, show that

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k a^k b^{n-k}$$

where ${}^nC_k = \frac{n!}{k!(n-k)!}$. You may use ${}^{n+1}C_k = {}^nC_{k-1} + {}^nC_k$ ($1 \leq k \leq n$). (The binomial theorem was known to the Arabs. They did not have a complete proof, but after verifying it for few small n they stated the general form using in a quasi-inductive argument. Later the theorem was rediscovered by Isaac Newton (1654-1705), who included it in his celebrated *Philosophiae Naturalis Principia Mathematica* (1687). For the proof he used a combinatorial argument. The first inductive proof was by Jakob Bernoulli (1654-1705), published posthumously in his *Ars Conjectandi* (1713)).

Problem 2.9. It is easy to see that the number $(2+\sqrt{3})^n$ can be written in the form $a_n + b_n\sqrt{3}$. Show a) inductively and b) without induction, that the numbers a_n, b_n satisfy $a_n^2 - 3b_n^2 = 1$ ($n\in\mathbf{N}$).

Problem 2.10. Show that the number $2^{2^n} - 1$ is divisible by at least n distinct primes.

Problem 2.11. If $F_n = a^{2^n} + 1$ is the n^{th} Fermat number ($n = 0, 1, 2, \ldots$), show that

$F_n - 2 = (a - 1)F_0F_1\ldots F_{n-1}$ ($n\in\mathbf{N}$).

Problem 2.12. Prove by induction that $n! > 3^n$ for $n \geq 7$.

Problem 2.13. If a_0, a_1, a_2, … is a sequence of positive real numbers satisfying $a_0 = 1$ and $a_{n+1}^2 > a_n a_{n+2}$ (n = 0, 1, 2, …), show that $a_1 > a_2^{1/2} > a_3^{1/3} > a_4^{1/4} > ... > a_n^{1/n} > ...$.

Problem 2.14. A result of Ramanujan (whose proof is beyond the scope of this book) states that $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+...}}}}}$ = 3. Use Ramanujan's result to show that for all n$\in$**N,** $\sqrt{1+n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+...}}}}} = n+1.$

Section 3. Patterns

One of the disadvantages of the method of induction, as reflected by some of the examples portrayed above (especially in Problem 1), is that one needs to know *beforehand* the formula describing the situation considered. It is only then that one may embark on proving it. But this need for foreknowledge can often be remedied by detecting patterns after judicial evaluation of special cases. In practice it means that one needs to *conjecture* the underlying rule, and then verify whether it is, indeed, correct. In other words, we have to do some guessing. The following examples elucidate this point.

Example 3.1. For what values on n is $2^n + 1$ a multiple of 3?

Solution. By checking small values of the integer n one realizes that $2^n + 1$ is a multiple of 3 for n equals 1, 3, 5 and 7, but fails to be so when n equals 2, 4, 6 or 8. It seems reasonable to guess that $2^n + 1$ a multiple of 3 precisely when n is odd. This turns out to be correct, and the following inductive argument can be used (how?) to verify the claim: Write $a_n = 2^n + 1$. Then $a_{n+2} = 2^{n+2} + 1 = 4(2^n + 1) – 3 = 4a_n – 3$, which is a multiple of 3 precisely when a_n is. □

Example 3.2. If f(x) = 2x + 1, guess a formula for the nth term of the sequence f_1 = f(x), f_2 = f(f(x)), f_3 = f(f(f(x))), f_4 = f(f(f(f(x)))), … and then prove it by induction.

Solution. By direct calculation one verifies that $f_2 = 4x + 3$, $f_3 = 8x + 7$, $f_4 = 16x + 15$ and so on. If these examples are not adequate to guess the pattern, the reader should continue with further iterations of f. Sooner or later one suspects that $f_n = 2^n x + 2^n - 1$. It turns out that the guess is correct, as the reader should supply the missing portions of the following inductive argument that settles the matter: $f_{n+1} = f(f_n(x)) = f(2^n x + 2^n - 1) = 2(2^n x + 2^n - 1) + 1 = 2^{n+1}x + 2^{n+1} - 1$. □

Example 3.3. By considering the numerical sequence

2 – 1, 3 – (2 – 1), 4 – (3 – (2 – 1)), 5 – (4 – (3 – (2 – 1))), …

guess and then prove inductively the numerical value of

n – (n – 1 – (n – 2 – (n – 3 – (…– (3 – (2 – 1))...))).

Solution. The first few expressions simplify to 1, 2, 2, 3, 3 and 4 respectively. One may guess that the general pattern is

$$n - (n - 1 - (n - 2 - (n - 3 - (\ldots - (3 - (2 - 1))...))) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

This is easy to verify inductively and the details are left to the reader, who should consider separately the cases n even and n odd. □

A word of caution is necessary here: No matter how many initial cases we check in a particular situation, a pattern that seems to emerge is not sufficient to draw conclusions. A proof must *always* follow our guess and failure to devise such a proof may indicate that our conjecture is, perhaps, wrong. There are several examples showing that even first rate mathematicians were deceived by a few special cases. The great Fermat, for example, after observing that $2^{2^0}+1=3$, $2^{2^1}+1=5$, $2^{2^2}+1=17$, $2^{2^3}+1=257$ and $2^{2^4}+1=65537$ are prime numbers, thought that $2^{2^n}+1$ is a prime for each n. This turned out to be false, and the first counterexample was given by Euler who found that $2^{2^5}+1=641\times 6700417.$

Sometimes the first counterexample to what might appear to be a pattern is very far away. For instance, the numbers http://primes.utm.edu/glossary/page.php/GCD.html$n^{17}+9$ and $(n+1)^{17}+9$ are relatively prime for n =1, 2, 3, . . . successively, and for a *very* long time after that. But is this always the case? No, and the first counterexample is for

n = 8424432925592889329288197322308900672459420460792433.

There are two delightful articles by Richard Guy, entitled *The Strong Law of Small Numbers* (American Mathematical Monthly, (1988) 697-711) and *The Second Strong Law of Small Numbers* (Mathematics Magazine, 63 (1990) 3 - 20) with numerous examples of sequences that *seem* to follow a pattern. But in some cases the reality is, against all intuition, very different. It is worth also looking at the web page

http://primes.utm.edu/glossary/page.php?sort=LawOfSmall

where the previous example appears.

Here are some problems along the above lines, where the reader is invited either (i) to discover a pattern and then prove his/her hypothesis correct, or (ii) to find a counterexample that contravenes the pattern that appears at first sight.

Problem 3.1. After guessing an appropriate formula by testing a few first values of n, use an inductive argument to find the following sums.

1. $1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1} n^2$,
2. $1 \cdot (1!) + 2 \cdot (2!) + 3 \cdot (3!) + \ldots + n \cdot (n!)$,
3. $n^2 - [(n-1)^2 - [(n-2)^2 - [(n-3)^2 - [\ldots - [3^2 - (2^2 - 1^2)]\ldots]]]]$,
4. $\dfrac{1}{x(x+1)}+\dfrac{1}{(x+1)(x+2)}+\dfrac{1}{(x+2)(x+3)}+\ldots+\dfrac{1}{(x+n-1)(x+n)}$.

Problem 3.2. It is given that the sum $1^6 + 2^6 + 3^6 + \ldots + n^6$ can be simplified in the form $n(n + 1)(2n + 1)(An^4 + Bn^3 - 3n + 1)/42$, where A and B are constants independent of n. Guess appropriate values of A and B and then verify that they lead to a valid formula.

Problem 3.3. If (p_n) is the sequence of primes starting from $p_1 = 2$, show that the sequence of numbers $p_1 + 1$, $p_1 p_2 + 1$, $p_1 p_2 p_3 + 1$, ... , $p_1 p_2 p_3 \ldots p_n + 1$, used by Euclid in a proof in his *Elements*, consists of prime numbers for n = 1, 2, 3, 4, 5 but not for n = 6.

Problem 3.4. Given n points on the circumference of a circle, where n is successively 1, 2, 3, 4,... , draw (in separate figures) all chords joining them. For this make sure that the points are "in general position" in the sense that no three chords are concurrent. Now, count the regions into which each circle is partitioned by the chords. You will find that they are, successively 1, 2, 4, 8, 16, ... What pattern seems to emerge? Is the next answer 32? Show that it is not!

Problem 3.5. Guess the general term of the sequence (a_n) if $a_0 = 1$, $a_1 = 2$ and for $n \geq 1$, $a_{n+1} = \sqrt{a_n + 6\sqrt{a_{n-1}}}$.

Problem 3.6. Guess the general term of the sequence (a_n) if $a_1 = 1$, and for $n \geq 2$ we have $\sqrt{a_1} + \sqrt{a_2} + ... + \sqrt{a_n} = \frac{1}{2}(n+1)\sqrt{a_n}$.

Section 4. Divisibility

The method of induction can be applied to an abundance of situations, not just proving formulae as, perhaps, most of the above examples suggest. In what follows we shall see some of these different circumstances. We start with a fairly easy situation, the case of divisibility of integers, of which we have already seen some problems in Section 2. We shall use the notation a | b to signify that an integer a divides, or is a factor of, an integer b.

Example 4.1. Show that for each positive integer n we have $9 \mid 5^{2n} + 3n - 1$; that is, 9 divides the number $5^{2n} + 3n - 1$.

Solution. Let $a_n = 5^{2n} + 3n - 1$. It is clear that $a_1 = 27$ is divisible by 9. Assume now that for n = k, the number a_n is divisible by 9, that is, $5^{2k} + 3k - 1 = 9M$ for some integer M. We have to show that $a_{k+1} = 5^{2(k+1)} + 3(k+1) - 1 = 25.5^{2k} + 3k + 2$ is also divisible by 9. The idea is to somehow use our inductive hypothesis, and this can be done as follows:

$$\begin{aligned} a_{k+1} &= 25 \cdot 5^{2k} + 3k + 2 \\ &= 25 \cdot (5^{2k} + 3k - 1) - 72k + 27 \\ &= 25 \cdot 9M - 9(8n - 3) \quad \text{(by the inductive hypothesis)} \end{aligned}$$

i.e. a_{k+1} is a multiple of 9.

Therefore by the principle of induction $9 \mid a_n$ for all positive integers n.□

Problem 4.1. Redo the previous example more elegantly by considering $a_{k+1} - 25a_k$ in place of a_{k+1} alone.

Example 4.2. Show that all numbers in the sequence 1003, 10013, 100113, 1001113,… and so on, are divisible by 17.

Solution. We have 1003 = 17×59, moreover, the difference between two consecutive numbers of the sequence is of the form 9010…0, which is also a multiple of 17 (note 901 = 17×53). With this information the reader should be able to fill the details of a full inductive argument.□

Problem 4.2. Show that for each $n \in \mathbf{N}$, $7^{2n} - 48n - 1$ is a multiple of 2304.

Problem 4.3. Show that for each $n \in \mathbf{N}$, $3 \cdot 5^{2n+1} + 2^{3n+1}$ is a multiple of 17.

Problem 4.4. Show that the sum of cubes of any three consecutive integers is divisible by 9.

Section 5. Inequalities.

We have seen an inequality, Bernoulli's inequality (Example 2.1), that depends on a natural number n. This particular one was proved using induction and, sure enough, many inequalities that depend on n can be dealt with by induction. For instance the following generalisation of Bernoulli's inequality can be shown by a minor modification of the proof given above.

Example 5.1 (Weierstrass inequality). If a_n ($n \in N$) are real numbers that are either all positive or all in [-1, 0] then

$$\prod_{k=1}^{n}(1+a_k) \geq 1+\sum_{k=1}^{n} a_k$$

Proof. As mentioned, the proof follows closely that of Bernoulli's inequality given above, and the details are left to the reader: For the inductive step then one only needs to multiply both sides by the positive number $(1 + a_{n+1})$, but some care must be taken when all a_n are in [-1, 0], in which case the term involving the summation sign is negative but $a_{n+1}\left(\sum_{k=1}^{n} a_k\right)$ positive. □

There are several inequalities in the text and in the problems of what follows, but here is a preliminary set.

Problem 5.1. Prove by induction that a) $2^n > n^2$ for $n \geq 5$, b) $2^n > n^3$ for $n \geq 10$.

Problem 5.2. Prove by induction that $2!4!\ldots(2n)! > [(n + 1)!]^n$ ($n \in \mathbf{N}$).

Problem 5.3. Prove that $(2n)!(n + 1) > 4^n(n!)^2$ for all $n > 1$.

Problem 5.4. Prove for all integers $n > 1$ the inequality

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}} > 2\sqrt{n+1}-2.$$

Problem 5.5. Prove that if a_k satisfies $0 < a_k < 1$ for $1 \leq k \leq n$, then

$$(1 - a_1)(1 - a_2)\ldots(1 - a_n) > 1 - (a_1 + a_2 + \ldots + a_n).$$

Problem 5.6. Prove that if a_k satisfies $0 \leq a_k \leq 1$ for $1 \leq k \leq n$, then

$$2^{n-1}(1 + a_1 a_2 \ldots a_n) \geq (1 + a_1)(1 + a_2)\ldots(1 + a_n).$$

Section 6. Variations of induction

Up to now the proof of a statement P(n) for all positive integers n, proceeded by verifying P(1) and then P(k +1) from the assumption that P(k) is true. There are variants of the inductive argument as will be shown in the following paragraphs.

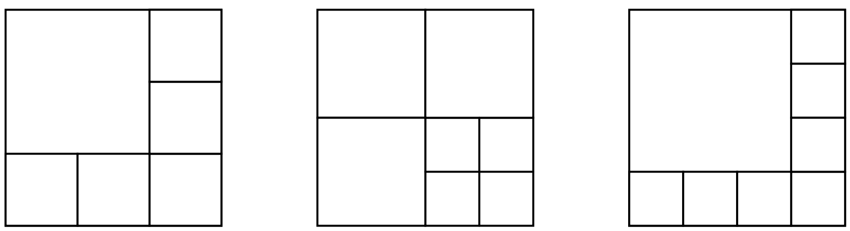
a) Jumps: In this method we prove the validity of a statement P(n) by proceeding, say, 2 steps at a time. In other words, the inductive step establishes the validity of P(k + 2) from the assumption that P(k) is true. If, in addition, we verify that P(1) and P(2) are true, then we reach our goal as we clearly have the implications $P(1) \Rightarrow P(3) \Rightarrow P(5) \Rightarrow P(7) \Rightarrow \ldots$ and $P(2) \Rightarrow P(4) \Rightarrow P(6) \Rightarrow P(8) \Rightarrow \ldots$ which, collectively, cover all cases of P(n). Similarly we may proceed in jumps of any fixed $t \in \mathbf{N}$. This requires showing the validity of $P(k) \Rightarrow P(k + t)$ and of P(1), P(2), …, P(t).

Example 6.1. Show that each $n \in \mathbf{N}$, the equation $a^2 + b^2 = c^n$ has a solution in positive integers.

Solution. We work using jumps of 2: The cases for n = 1 and 2 are clear. Now, if $a_1^2 + b_1^2 = c_1^k$ is a particular positive integer solution of the given equation for n = k, then a solution for the case n = k + 2 is obtained from $(c_1 a_1)^2 + (c_1 b_1)^2 = c_1^{k+2}$. □

Example 6.2. It is clear that a square can be divided into subsquares by drawing segments parallel to the sides. Show that it can be divided into n squares (of not necessarily equal size) whenever $n \geq 6$.

Solution. The figures below show how to divide the square into 6, 7 or 8 subsquares. Since a square can be further divided into four smaller ones, application of this operation increases the total number of squares in a subdivision by three (4 new ones and one lost). So we can use (how?) an inductive argument, jumping in 3's, to complete the proof. □



Note that the leaps need not be constant. Here is an example.

Example 6.3. Show that there exists an infinite number of triangular numbers that are perfect squares. (Recall, triangular numbers are the integers of the form $T_n = 1 + 2 + \ldots + n = \frac{1}{2} n(n + 1)$).

Solution. $T_1 = 1 = 1^2$. Suppose now that the triangular number T_k is a perfect square. Our problem is to utilize this information and find a bigger one that is also a perfect square. Somehow T_{k+1}, T_{k+2} etc. do not seem to work and we need to do better than that. A moment's reflection gives us a better choice: $T_{4k(k+1)} = 4k(k + 1)[4k(k + 1) + 1]/2 = 4(4k^2 + 4k + 1)\, T_k = 4(2k + 1)^2 T_k$ is clearly a perfect square along with T_k. □

Problem 6.1. Use an inductive argument in jumps of 2 to show that for all $n \in \mathbf{N}$,

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = (-1)^{n-1}(1 + 2 + \ldots + n).$$

Problem 6.2. (Eötvös Competition 1901). Use an inductive argument in jumps of 4 to show that $1^n + 2^n + 3^n + 4^n$ is divisible by 5, if and only if n is not divisible by 4.

Problem 6.3. Use an inductive argument in jumps of 3 to show that no number of the form $2^n + 1$ is a multiple of 7.

Problem 6.4. After verifying the simple equations $\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{6^2} = 1$, $\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} = 1$ and $\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{14^2} + \frac{1}{21^2} = 1$, show using an inductive argument with jumps of 3 that for every $n \geq 6$ there exist integers $a_1, a_2, \dots, a_n$ such that $\frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2} = 1$.

Problem 6.5. (Erdös-Suranyi theorem). After verifying the simple equations $1 = 1^2$, $2 = -1^2 - 2^2 - 3^2 + 4^2$, $3 = -1^2 + 2^2$ and $4 = -1^2 - 2^2 + 3^2$ show that for each natural number N there is an n and an appropriate choice of + and - signs (which we write as $\pm$ in short) such that $N = \pm 1^2 \pm 2^2 \pm 3^2 \pm \dots \pm n^2$.

b) Strong induction: In Euclid's *Elements* it is shown that every integer $k > 1$ is a product of (one or more) primes. His proof is essentially the following. The statement is clearly true for $k = 2$. Suppose now that we have proved that all integers *up to and including* k are products of primes. Then for k+1 we can argue that it is either a prime number, in which case we are done, or a product of two smaller numbers. But each of these two smaller numbers are, by assumption, products of primes and hence so is k+1. By iterating the argument we conclude the corresponding property for k+2, k+3, etc, and eventually for all integers.

In other words, Euclid's argument is a stronger version of induction where a) we verify P(1) and b) prove statement P(k+1) by assuming that *all* of P(1), P(2), … , P(k) are true (not just the last one P(k)). The inductive schema we invoke is then the validity of the implications $P(1) \Rightarrow [P(1) \text{ and } P(2)] \Rightarrow [P(1) \text{ and } P(2) \text{ and } P(3)] \Rightarrow [P(1) \text{ and } P(2) \text{ and } P(3) \text{ and } P(4)]$, and so on, finally covering all P(n).

A simpler version of strong induction is to prove P(k+1) from the validity of P(k-1) and P(k) (and not utilizing still smaller integers). In other words we first verify the validity of P(1), P(2) and then of the implication $[P(k-1) \text{ and } P(k)] \Rightarrow [P(k+1)$. Note that here we proceed, essentially, by the steps $[P(1) \text{ and } P(2)] \Rightarrow [P(2) \text{ and } P(3)] \Rightarrow [P(3) \text{ and } P(4)]$, and so on.

Of course there are further variations, such as proving P(k+1) from the validity of P(k-2), P(k-1) and P(k), after verifying the statement for small n.

The following paradigms exemplify these ideas.

Example 6.4. A sequence (a_n) satisfies $a_1 = a_2 = 4$ and $a_{n+1}a_{n-1} = (a_n - 6)(a_n - 12)$ for $n = 2, 3, \dots$. Show that it is constant.

Solution. Of course we expect the constant to be 4, the common value of a_1 and a_2. We assume then that $a_{k-1} = a_k = 4$. Using now *both* these assumptions, we conclude from the recursion that $4a_{k+1} = (4 - 6)(4 - 12) = 16$, so that $a_{k+1} = 4$. Since by assumption $a_1 = a_2 = 4$, it is easily seen that, for all n, we have $a_n = 4$. □

Example 6.5. Recall that the natural numbers satisfy $\sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2$ for all $n \in \mathbf{N}$. Show that, conversely, if $a_n > 0$ is a sequence of real numbers such that $\sum_{k=1}^{n} a_k^3 = (\sum_{k=1}^{n} a_k)^2$ for all $n \in \mathbf{N}$, then $a_n = n$ ($n \in \mathbf{N}$).

Solution. The case $n = 1$ gives $a_1^3 = a_1^2$, so that $a_1 = 1$ (as $a_n > 0$). Assume now that *for all values of k up to m* we have $a_k = k$, in other words $a_1 = 1, \dots, a_m = m$. This is our strong inductive hypothesis and we are going to use every bit of it. For n = m + 1 we have, by assumption,

$$1^3 + 2^3 + ... + m^3 + a_{m+1}^3 = (1 + 2 + ... + m + a_{m+1})^2$$

$$= (1 + 2 + ... + m)^2 + 2(1 + 2 + ... + m)a_{m+1} + a_{m+1}^2$$

so clearly $a_{m+1}^3 = m(m + 1)a_{m+1} + a_{m+1}^2$ and so $a_{m+1}(a_{m+1} + m)(a_{m+1} - m - 1) = 0$, from which the claim follows. □

Problem 6.6. For the Fibonacci sequence, defined by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, show that a) $F_nF_{n+1} - F_{n-2}F_{n-1} = F_{2n-1}$, b) $F_{n+1}F_{n+2} - F_nF_{n+3} = (-1)^n$.

Problem 6.7. Let (a_n) be the sequence of Example 6.4 with the only difference that $a_1 = 2$ and $a_2 = 20$. Show that $a_n = 9n^2 - 9n + 2$ ($n\in\mathbf{N}$). If, instead, we had $a_1 = 2$ and $a_2 = 5$, show $a_n = 4 + (-\frac{1}{2})^{n-2}$.

Problem 6.8. Given an angle a, define x by $x + 1/x = 2\cos a$. Show that $x^n + 1/x^n = 2\cos na$ ($n\in\mathbf{N}$).

Problem 6.9. Show that if a, b satisfy a + b = 6 and ab = 1, then the number $a^n + b^n$ a) is always an integer and b) is never divisible by 5.

Problem 6.10. Let $a_n = (3+\sqrt{5})^n + (3-\sqrt{5})^n$. Show that a_n is an integer and that $2^n | a_n$.

Problem 6.11. Prove the statement of Pascal in his *Traité du Triangle Arithmetique* referred to in Section 1 above.

Problem 6.12. Let $a_1, a_2, a_3, \ldots$ be positive integers chosen such that $a_1 = 1$ and $a_n < a_{n+1} \le 2a_n$ ($n\in\mathbf{N}$). Show that every positive integer can be written as a sum of distinct a_n's.

Problem 6.13. Let a sequence (b_n) satisfy $b_1 = 1$, $b_{2n} = b_n$ and $b_{2n+1} = b_{2n} + 1$. Show that b_n equals the number of ones in the binary representation of n.

c) Double induction: There are cases where the proof of the inductive step requires, in its own right, an inductive argument. The following examples illustrate this point.

Example 6.6. Show that for each $n\in\mathbf{N}$, $2 \cdot 7^n + 3 \cdot 5^n - 5$ is a multiple of 24.

Solution. Writing $a_n = 2 \cdot 7^n + 3 \cdot 5^n - 5$, the claim is clear for n = 1. Assuming it true for n = k then as $a_{k+1} = 7 \cdot a_k - 6 \cdot 5^k + 30$, the inductive argument would be complete if we could prove that $6 \cdot 5^k - 30$ is always a multiple of 24. We can now start a new inductive argument to prove the last statement, an easy task left to the reader. □

d) Two dimensional induction: So far we have considered statements P(n) depending on a single integer n. But sometimes we meet statements depending on two (or more) integers. A useful inductive way to deal with such a statement, for simplicity call it P(m,n), is to proceed in stages, intermingling the m's and n's. For instance, we can prove the validity of a) P(1,1), then b) of P(2,1) and P(1,2), then c) of P(3,1), P(2,2) and P(1,3), and so on. This particular description moves, so to speak, 'diagonally'. Any other way which covers all (m,n) in stages, is just as acceptable.

Example 6.7. (IMO 1972). Prove that (2m)!(2n)! is a multiple of m!n!(m+n)! for any non-negative integers m and n.

Solution. We are to show that C(m, n) = (2m)!(2n)!/(m!n!(m+n)!), for m, n $\geq$ 0, is integral. This is certainly the case for C(m, 0) = (2m)!/(m!m!) (we leave this to the reader: one way to see it is to recognize it as a binomial coefficient). Finally it is easy to verify that C(m, n) = 4C(m, n – 1) - C(m + 1, n – 1), from which, using the previous, we can in turn verify that C(m,1) is integral for all m, then C(m, 2) for all m, C(m, 3) for all m, and so on. □

Problem 6.14. Prove inductively that the product of r consecutive integers is divisible by r!

Problem 6.15. If (F_n) denotes the Fibonacci sequence, prove that $F_n^2 + F_{n+1}^2 = F_{2n+1}$ and $2F_nF_{n+1} + F_{n+1}^2 = F_{2n+2}$. (Hint: Let P(n) be the first identity and Q(n) the second. Induction proceeds via P(1) $\Rightarrow$ Q(1) $\Rightarrow$ P(2) $\Rightarrow$ Q(2) $\Rightarrow$ P(3) $\Rightarrow$ ...).

e) Back and forth: This variant of the usual inductive procedure is in two steps. First one shows P(1) $\Rightarrow$ P(n_1) $\Rightarrow$ P(n_2) $\Rightarrow$ P(n_3) $\Rightarrow$ … for some chosen but fixed sequence 1 < n_1 < n_2 < n_3 < … . Then shows the *backward* step P(k) $\Rightarrow$ P(k-1). A moment's reflection shows that the backward step fills the gaps between the numbers 1, n_1 , n_2 , n_3 , … left unattended in the first step, completing the proof. Here is an example of such a proof of the AM-GM inequality. The first step uses the sequence 1 < 2 < 2^2 < 2^3 < … .

Example 6.8. Show that for any sequence (a_n) of positive numbers we have

$$\left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^n \geq a_1 a_2 \cdot ... \cdot a_n \qquad (n \in \mathbf{N})$$

Solution. The case for n = 1 is trivial. Assuming validity of P(k) for all sequences (a_n) of positive numbers, verification of P(2k) is as follows: Apply P(k) to the sequence $((a_{2n-1} + a_{2n})/2)$. We get

$$\left(\frac{\frac{a_1 + a_2}{2} + \frac{a_3 + a_4}{2} + ... + \frac{a_{2k-1} + a_{2k}}{2}}{k}\right)^k \geq \frac{a_1 + a_2}{2} \cdot \frac{a_3 + a_4}{2} \cdot ... \cdot \frac{a_{2k-1} + a_{2k}}{2}$$

$$\geq \sqrt{a_1 a_2} \cdot \sqrt{a_3 a_4} \cdot ... \cdot \sqrt{a_{2k-1} a_{2k}}$$

which is easily rewritten as P(2k), namely

$$\left(\frac{a_1 + a_2 + ... + a_{2k}}{2k}\right)^{2k} \geq a_1 a_2 \cdot ... \cdot a_{2k}.$$

So we now know the inequality for the cases P(1), P(2), P(2^2), P(2^3), … .

For the backward step, we assume P(k) and derive P(k-1). For this purpose, by P(k) applied to the k numbers a_1, a_2, …, a_{k-1} and $(a_1 + a_2 + … + a_{k-1})/(k-1)$, we have

$$\left(\frac{a_1+a_2+\ldots+a_{k-1}+\dfrac{a_1+a_2+\ldots+a_{k-1}}{k-1}}{k}\right)^k \geq a_1a_2\cdot\ldots\cdot a_{k-1}\cdot\frac{a_1+a_2+\ldots+a_{k-1}}{k-1}$$

which is easily seen to reduce to statement P(k-1) after collecting terms. □

Problem 6.16. Redo the step P(k) ⇒ P(2k) in the proof of Example 6.7 by making use of the equality $\sqrt[2k]{a_1a_2\cdot\ldots\cdot a_{2k}} = \sqrt{\sqrt[k]{a_1a_2\cdot\ldots\cdot a_k}\cdot\sqrt[k]{a_{k+1}a_{k+2}\cdot\ldots\cdot a_{2k}}}$, thereby giving a slightly different proof of the AM-GM inequality.

Problem 6.17. (Jensen's inequality). If f : I → **R**, where I ⊆ **R** is an interval, is a concave function, then $f\left(\frac{a_1+a_2+\ldots+a_n}{n}\right) \geq \frac{f(a_1)+f(a_2)+\ldots+f(a_n)}{n}$ for all $a_1, a_2, \ldots, a_n$ in I. The reverse inequality is true for convex functions. (Recall, concave functions satisfy $f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$ and convex ones the reverse inequality).

f) Strengthening: The following example illustrates this curious technique, which will be explained immediately after.

Example 6.9. Show that for n ≥ 2, $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2} \leq \frac{3}{4}$.

Solution. Here the inductive step does not work, so will modify our approach. We show instead the *stronger* inequality P(n): $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2} \leq \frac{3}{4}-\frac{1}{n}$.

The case n = 2 is immediate, and for the inductive step we can clearly argue along the lines $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{k^2}+\frac{1}{(k+1)^2} \leq \frac{3}{4}-\frac{1}{k}+\frac{1}{(k+1)^2} \leq \frac{3}{4}-\frac{1}{k+1}$, which completes the proof. □

The curiosity is that although we failed to prove a statement, we managed to prove a stronger one! The mystery clarifies if we realize that proof of the inductive step the second time was based on a *stronger hypothesis*. So it is not surprising that the conclusion was also stronger. In the failed attempt, the inductive hypothesis was too weak to prove the full statement.

Problem 6.18. Prove the inequality $\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\ldots\cdot\frac{(2n-1)^2}{(2n)^2} < \frac{1}{3n}$.

Problem 6.19. Prove the inequality $\frac{1}{2\sqrt{1}}+\frac{1}{3\sqrt{2}}+\ldots+\frac{1}{(n+1)\sqrt{n}} < 2$.

Problem 6.20. Show that $(1+\frac{1}{2^3})(1+\frac{1}{3^3})\cdot\ldots\cdot(1+\frac{1}{n^3}) \le 3.$

The technique of strengthening has so far been used only to prove inequalities. One should not draw the conclusion that this is the only place that it can be used. Here is an example.

Example 6.10. Show that for every n there exist n distinct divisors of n! whose sum is n!

Solution. Before enunciating which exactly is the strengthened statement, let us attempt an inductive proof: The case n = 1 is clear. Assuming that there are k distinct divisors d_1, d_2, …, d_k of k! whose sum is k!, we seek k + 1 distinct divisors of (k + 1)! whose sum is (k + 1)!. Consider $(k + 1)d_1$, $(k + 1)d_2$, … , $(k + 1)d_k$. They are divisors of (k + 1)!, they are distinct, they sum up to (k + 1)! but the problem is that they are only k of them. If we replace $(k + 1)d_1$ by kd_1 and d_1, we have k + 1 numbers but now one of them, namely kd_1, may not be a divisor of (k + 1)!. There is a way out of this difficulty, and this is by taking $d_1 = 1$, but are we allowed to do this? The answer is yes if we start all over again, but this time we strengthen our original statement to showing that "for every n there exist n distinct divisors of n! whose sum is n! *and such that one of the divisors is 1*". The procedure is now clear and the details are left to the reader. □

Section 7. Subtleties

At the beginning of this chapter we talked about the versatility of induction as a proving device. With the examples below we will see more clearly the diversity and adaptability of the versatile tool we are discussing. Here the application of the inductive hypothesis will be slightly more intricate.

Before coming to the first example, recall that so far the variable n to which we applied an inductive argument was pretty clear from the premises of the problem. There are, however, some interesting cases where the choice of the variable with which we chose to work, is rather subtle.

Example 7.1. Show that for any set $\{a_1, a_2,\ldots, a_n\}$ of nonnegative integers, the expression $X = \frac{(a_1+a_2+\ldots+a_n)!}{a_1!a_2!\ldots a_n!}$ is an integer.

Solution. Induction will not be on n but rather on the number $N = a_1 + a_2 +\ldots+ a_n$. If N = 1, in which case (without loss of generality) $a_1 = 1$, $a_2 = \ldots = a_n = 0$, the result is trivial. Suppose now that for some $k \ge 1$, X is an integer whenever the sum $a_1 + a_2 +\ldots+ a_n = k$. We show the same thing for n nonnegative integers whose sum is k + 1. Note that we may assume that $a_j \ge 1$ for all $1 \le j \le n$ (if some $a_j = 0$, it gives no contribution in X, so we may delete it).

Let then $a_1 + a_2 +\ldots+ a_n = k + 1$. By the inductive hypothesis applied to the numbers $a_1 - 1, a_2, \ldots , a_n$, we have that

$$\frac{a_1X}{a_1+a_2+\ldots+a_n} = \frac{(a_1-1+a_2+\ldots+a_n)!}{(a_1-1)!a_2!\ldots a_n!}$$

is an integer. Similarly $a_2X/(a_1 + a_2 +\ldots+ a_n)$, … , $a_nX/(a_1 + a_2 +\ldots+ a_n)$ are also integers, and hence so are their sum

$$X = \sum_{j=1}^{n} \frac{a_j X}{a_1 + a_2 + \ldots + a_n}. \ \square$$

Problem 7.1. Give another proof of Example 7.1 using the identity $\frac{(a+b+\ldots+c)!}{a!b!\ldots c!} = \frac{((a+b)+\ldots+c)!}{(a+b)!\ldots c!} \times \frac{(a+b)!}{a!b!}$.

In the next two examples we apply our inductive hypothesis in a more dexterous way.

Example 7.2. Let A be any subset of {1, 2, 3, ... , 2n - 1} with n elements, where $n \in \mathbf{N}$. Show that there are elements x and y of A (not necessarily distinct) with x + y = 2n.

Solution. For n = 1 the result is clear. Assume the conclusion true for n = k and consider now a subset A of {1, 2, 3, ... , 2k + 1} with k + 1 elements. We are to show that there exist x and y in A with x + y = 2(k + 1). If 1 and 2k + 1 are both in A, we are done, so we may assume that at least one of the two is missing. Delete from A the other. What remains is a set A' of at least k elements such that A' $\subseteq$ {2, 3, ... , 2k}. Subtract 1 from each element of A', to get a subset of {1, 2, 3, ... , 2k -1} with (at least) k elements. *We apply our inductive hypothesis to this last set*: Thus there are x and y in A such that (x - 1) + (y - 1) = 2k, and so x + y = 2(k + 1). $\square$

Problem 7.2. (Hermite's identity) If n is a positive integer and x a real number, prove that

$$[x] + [x + \frac{1}{n}] + [x + \frac{2}{n}] + \ldots + [x + \frac{n-1}{n}] = [nx],$$

where [.] denotes 'integer part'. (Hint: induction is not on n but rather on the unique $k \in N$ with $k/n \le x < (k+1)/n$).

Problem 7.3. There are n fuelling stations on a circular track and the total gas among them is just enough for a car to complete the circuit. Show that there is a fuelling station from which the car can start and manage to complete the circuit. The car is allowed to use only the gas provided at the fuelling stations, which it can collect only as it goes along.

Section 8. Harder Questions

Problem 8.1. If x is a real number not of the form n + ½ for an integer n, let {x} denote the nearest integer to x (so that for example {e} = {π} = 3). Show $\sum_{k=1}^{n^2+n} \{\sqrt{k}\} = 2\sum_{k=1}^{n} k^2$.

Problem 8.2. Let n be an integer. Consider all points (a,b) of the plane with integer co-ordinates such that $0 \le a$, $0 \le b$, $a + b \le n$. Show that if these points are covered by straight lines then there are at least n + 1 such lines.

Problem 8.3. Given a positive integer N perform the following operation to obtain a new integer s(N): First write N in its decimal form as N = $\overline{a_n a_{n-1} \ldots a_0}$ and then set s(N) =

$\sum a_k^2$. Show that repeated application of this operation will eventually lead to the number 1 or to the cycle 4, 16, 37, 58, 89, 145, 42, 20. (Remark: One can check by hand the validity of the claim for all three figure numbers, a fact which you may take for granted. Induction on N starts thereafter.)

Problem 8.4. Show that every member of the sequence defined by $a_1 = a_2 = a_3 = 1$ and

$a_{n+3} = (1 + a_{n+1}a_{n+2})/a_n$ ($n \geq 1$) is an integer.

Problem 8.5. If m and n are positive integers, show that so is $\dfrac{(mn)!}{m!(n!)^m}$.

Problem 8.6 (Chebychev inequality). Let $a_1 \geq a_2 \geq ... \geq a_n \geq 0$ and $b_1 \geq b_2 \geq ... \geq b_n \geq 0$. Show that $\frac{1}{n}(\sum_{k=1}^{n} a_k).\frac{1}{n}(\sum_{k=1}^{n} b_k) \leq \frac{1}{n}(\sum_{k=1}^{n} a_k b_k)$. What is the corresponding inequality if $0 \leq a_1 \leq a_2 \leq ... \leq a_n$ and $0 \leq b_1 \leq b_2 \leq ... \leq b_n$?

Problem 8.7. (Putnam 1968, slightly differently worded). Let S be a set of n elements and let P be the set of all subsets of S. Show that we can label the elements of P as A_1, A_2, ... , A_{2^n} so that $A_1 = \varnothing$ and such that any two consecutive sets in this labeling differ by exactly one element of S.

Problem 8.8. (Putnam 1956, slightly differently worded). Given any 2n points ($n \geq 2$) that are joined by $n^2 + 1$ segments, show that at least one triangle is formed from these segments.

Problem 8.9. Let A be a subset of {1, 2, … , 2n} with n + 1 elements. Show inductively that there exist x, y in A such that x divides y.

Problem 8.10. Show that for any n > 1 there exists a finite set A_n of points on the plane such that for any $x \in A_n$ there are points $x_1, x_2, \ldots, x_n$ in A_n each of which is at a distance 1 from x.

Problem 8.11. (Adapted from IMO 1997). Show that there exist infinitely many values of n for which we can find an $n \times n$ matrix whose entries come from the set S = {1, 2, ... , 2n-1} and, for each k = 1, 2, ... , n, its k^{th} row and k^{th} column together contain all elements of S.

Solutions

2.1) All the exercises are routine. Note however that j) requires sin(2y) = 2sinycosy, k) requires $\sin\theta - \sin\phi = 2\sin\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)$ with θ = (2n + 2)x and φ = 2nx. For the inductive step of l) we need $\sqrt{2+2\cos\theta} = 2\cos\left(\frac{\theta}{2}\right)$ usually written in the more familiar form $\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$.

2.2) The inductive step uses $a_{k+1} + 1 = 2a_k + 2 = 2(a_k + 1) = 2^k(a_1 + 1)$. The second case is just as routine.

2.3) Use $x_{n+1}y_n - x_ny_{n+1} = (a_{n+1} x_n + x_{n-1}) y_n - x_n(a_{n+1} y_n + y_{n-1}) = -(x_ny_{n-1} - x_{n-1}y_n)$.

2.4) Use $(x^2 + y^2)(u^2 + v^2) = (xu – yv)^2 + (xv + yu)^2$.

2.5) If $P(k) = 2k^5/5 + k^4/2 – 2k^3/3 – 7k/30$ then, expanding, $P(k+1) = P(k)$ + integer.

2.6) Use $x^{n+1} – y^{n+1} = x(x^n – y^n) + y^n(x – y)$.

2.7) It is easy to see that an addition of new vertex to an k-gon increases the number of diagonals by k – 1 and ½ k(k – 3) + k – 1 = ½ (k + 1)(k – 2).

2.8) This is a standard textbook proof.

2.9) a) Use $(2+\sqrt{3})^{n+1} = (a_n + b_n\sqrt{3})(2+\sqrt{3}) = (2a_n + 3b_n) + (a_n + 2b_n)\sqrt{3}$ so that $a_{n+1} = 2a_n + 3b_n$ and $b_{n+1} = a_n + 2b_n$. It is easy now to show that $a_{n+1}^2 - 3b_{n+1}^2 = 1$. b) It is easy to see by the binomial theorem that $(2+\sqrt{3})^n = a_n - b_n\sqrt{3}$. Now use $(2+\sqrt{3})^n(2-\sqrt{3})^n = (4-3)^n = 1$.

2.10) Use $2^{2^{n+1}} - 1 = (2^{2^n} - 1)(2^{2^n} + 1)$. Note that $2^{2^n} - 1$ and $2^{2^n} + 1$ do not have common prime divisors as they are both odd numbers differing by 2.

2.11) The case n = 1 is clear. From the hypothesis $F_k – 2 = (a – 1)F_0F_1\ldots F_{k-1}$ we have $F_{k+1} – 2 = a^{2^{k+1}} - 1 = (a^{2^k} - 1)(a^{2^k} + 1) = (F_k – 2) F_k = (a – 1)F_0F_1\ldots F_{k-1} F_k$.

2.12) 7! = 5040 > 2187 = 3^7. If $k! > 3^k$ (where $k \geq 7$) then $(k+1)! = (k+1)(k!) > (k+1).3^k \geq 8.3^k > 3^{k+1}$.

2.13) The condition $a_1^2 > a_0a_2 = a_2$ gives the first inequality. Assuming $a_{k-1}^{1/(k-1)} > a_k^{1/k}$ we have $a_k^2 > a_{k-1}a_{k+1} > (a_k)^{(k-1)/k}a_{k+1}$, from which the result easily follows.

2.14) The case n = 1 is Ramanujan's result. For the inductive step, let

$\sqrt{1+k\sqrt{1+(k+1)\sqrt{1+(k+2)\sqrt{1+(k+3)\sqrt{1+...}}}}} = k+1$. Now square both sides, subtract 1 and divide by k. It gives the next case.

3.1) a) $(-1)^n(1 + 2 + ... + n) = (-1)^n n(n + 1)/2$

b) $(n + 1)!$

c) $1 + 2 + ... + n = n(n + 1)/2$

d) $n/[x(x+n)]$

3.2) A = 3, B = 6.

3.3) Initially we find the primes 3, 7, 31, 211, 2311 but for n = 6 the result is 30031 = 59 x 509.

3.4) The next number, corresponding to n = 6, is 31.

3.5) We find $a_2 = 2^{3/2}$, $a_3 = 2^{7/4}$, $a_4 = 2^{15/8}$ etc. One may guess and then easily verify by induction that $a_n = 2^{(2^n-1)/2^{n-1}}$.

3.6) It is easy to verify that $a_2 = 4$, $a_3 = 9$ etc. The guess $a_n = n^2$ is correct and can be verified by induction. A quick way to do this is to verify first that $\sqrt{a_{n+1}} = \frac{n+1}{n}\sqrt{a_n}$.

4.1) This essentially the previous example: $a_{k+1} - 25a_k = -9(8k - 3)$.

4.2) If $a_n = 7^{2n} - 48n - 1$, for the inductive step consider $a_{k+1} - 49\, a_k = 2304k$.

4.3) If $a_n = 3.5^{2n+1} + 2^{3n+1}$, the inductive step can be sorted by writing $a_{k+1} - 25a_k = -17.2^{2n+1}$. Alternatively, we could consider $a_{k+1} - 8a_k = 3.17.5^{2k+1}$.

4.4) If $a_k = k^3 + (k + 1)^3 + (k + 2)^3$, then $a_{k+1} - a_k = (k + 3)^3 - k^3 = 9(k^2 + 3k + 3)$.

5.1) a) $2^5 = 32 \geq 5^2$. If $2^k > k^2$ (where $k \geq 5$) then $2^{k+1} = 2.2^k > 2.k^2 = k^2 + k^2 \geq k^2 + 5k > k^2 + 2k + 1 = (k + 1)^2$. b) $2^{10} = 1024 > 10^3$. If $2^k > k^3$ (where $k \geq 10$) then $2^{k+1} = 2.2^k > 2.k^3 = k^3 + k^3 \geq k^3 + 10k^2 \geq k^3 + 3k^2 + 3k + 1 = (k + 1)^3$.

5.2) The inductive step amounts to showing $(k + 2)\dots(2k + 2) > (k + 2)^{k+1}$. This is clearly true since each of the k + 1 terms of the left hand side is > (k + 2).

5.3) If $(2k)!(k + 1) > 4^k(k!)^2$ then $(2k + 2)!(k + 2) = (2k + 2)(2k + 1)[(2k)!(k + 1)](k + 2)/(k + 1) > (2k + 2)(2k + 1)4^k(k!)^2 (k + 2)/(k + 1) = 4^{k+1}((k + 1)!)^2 (2k + 1)(k + 2)/[2(k + 1)^2] > 4^{k+1}((k + 1)!)^2$.

5.4) The inductive step amounts to showing $\sqrt{n}+\frac{1}{\sqrt{n+1}}>\sqrt{n+1}$, which is routine.

5.5) The argument is a trivial adaptation of that of Example 5.1 in the text.

5.6) For the induction step assume validity of the inequality for n = m and any sequence (a_k) with $0 \le a_k \le 1$ for $1 \le k \le m$. Let now n = m + 1 and consider a sequence (b_k) with $0 \le b_k \le 1$ for $1\le k \le m + 1$. Apply the inductive hypothesis to $a_1 = b_1, a_2 = b_2, \dots , a_{m-1} = b_{m-1}$ and $a_m = b_m b_{m+1}$. Thus $2^m(1 + b_1 b_2 \dots b_{m-1}(b_m b_{m+1})) \ge 2(1 + b_1)(1 + b_2)\dots(1 + b_{m-1}) (1 + b_m b_{m+1})$. The required result follows upon observing that

$2(1 + b_m b_{m+1}) \ge (1 + b_m)(1 + b_{m+1})$ (which is true as equivalent to the true statement

$(1 - b_m)(1 - b_{m+1}) \ge 0$.)

6.1) The inductive step in jumps of two requires adding to $1^2 – 2^2 + 3^2 – 4^2 + \dots +$

$(-1)^{k-1}k^2$ the quantity $(-1)^k (k + 1)^2 + (-1)^{k+1}(k + 2)^2 = (-1)^{k-1} [- (k+ 1)^2 + (k + 2)^2] =$

$(-1)^{k-1} [(k+ 1) + (k + 2)]$. It is easily checked that this added to the right hand side gives the expected result.

6.2) The cases n = 1, 2, 3, 4 are immediate, for instance 5 does not divide $1^4 + 2^4 + 3^4 + 4^4 = 354$. For the inductive step use $1^{k+4} + 2^{k+4} + 3^{k+4} + 4^{k+4} – (1^k + 2^k + 3^k + 4^k) = 15.2^k + 80.3^k + 255.4^k$ = (a multiple of 5).

6.3) Use $2^{n+3} + 1 = 8.2^n + 1 = 7.2^n + (2^n +1)$.

6.4) If the n numbers $(a_1, a_2, \dots , a_{n-1}, a_n)$ have the stated property, then clearly so do the n + 3 numbers $(a_1, a_2, \dots , a_{n-1}, 2a_n, 2a_n, 2a_n, 2a_n)$.

6.5) We go in jumps of 4 using $(n + 1)^2 - (n + 2)^2 - (n + 3)^2 + (n + 4)^2 = 4$. Note further that the result extends to all integers: This is clear for -N if it is true for N, and $0 = 1^2 + 2^2 + 3^2 - 4^2 - 5^2 - 6^2 + 7^2$.

6.6) Routine using the definition.

6.7) For a_1 and a_2 the result is clear. For the inductive step, assuming $a_{k-1} = 9(k-1)^2 - 9(k-1) + 2$ and $a_k = 9k^2 - 9k + 2$, it is easily seen that $a_{k+1} = 9(k+1)n^2 - 9(k+1) + 2$.

6.8) For the inductive step assume validity for both n = k and n = k - 1. Then use $x^{k+1} + 1/x^{k+1} = (x + 1/x)(x^k + 1/x^k) - (x^{k-1} + 1/x^{k-1}) = 4(\cos a)(\cos ka) - 2\cos(k-1)a$.

6.9) a) Use $a^{n+1} + b^{n+1} = (a^n + b^n)(a + b) - ab(a^{n-1} + b^{n-1}) = 6(a^n + b^n) - (a^{n-1} + b^{n-1})$. b) Iterating the previous we get

$a^{n+1} + b^{n+1} = 6[6(a^{n-1} + b^{n-1}) - (a^{n-2} + b^{n-2})] - (a^{n-1} + b^{n-1}) = (\text{multiple of } 5) - (a^{n-2} + b^{n-2})$.

6.10) Note that $a_{n+1} = 6a_n - 4a_{n-1}$ (a quick way to see this is to observe first that $a = 3+\sqrt{5}$ and $b = 3-\sqrt{5}$ satisfy the equation $x^2 = 6x - 4$). It also follows inductively that $a_{n+1} = 6.2^n.(\text{integer}) - 4.\ 2^{n-1}.(\text{integer}) = 2^{n+1}.(\text{integer})$.

6.11) For the inductive step assume the result true for n = m and all k < n. Then ${}^{m+1}C_k : {}^{m+1}C_{k+1} = ({}^mC_{k-1} + {}^mC_k) : ({}^mC_k + {}^mC_{k+1})$. Now divide both numerator and denominator by ${}^mC_k$, and use ${}^mC_{k-1} : {}^mC_k = k : [n - (k-1)]$, ${}^mC_k : {}^mC_{k+1} = (k+1) : (n-k)$.

6.12) For the induction step assume that every positive integer x with $x \le a_k$ can be written as distinct elements from $a_1, \ldots, a_{k-1}, a_k$. Then any y with $a_k < y \le a_{k+1}$ can be written as $y = a_k + x$ with $1 \le x \le a_{k+1} - a_k \le a_k$ and the induction hypothesis may be applied to x.

6.13) An inductive proof follows easily from the observations: The binary representation of 2n is the same as that of n but with a zero adjoined in the end. That of 2n + 1 is the same as that of 2n only the last figure changes from 0 to 1.

6.14) For fixed r consider P(n) = n(n + 1)(n + 2)...(n + r -1), a product of r consecutive integers. Then P(n + 1) - P(n) = r x (n + 1)(n + 2)...(n + r -1) = r times the product of r - 1 consecutive integers. In other words, we have to prove the same statement, only for the product of r - 1 consecutive integers. This can built inductively from r = 1.

6.15) Routine using the hint given in the text.

6.16) The P(k) $\Rightarrow$ P(2k) step becomes

$$\sqrt[2k]{a_1a_2\cdot\ldots\cdot a_{2k}} = \sqrt{\sqrt[k]{a_1a_2\cdot\ldots\cdot a_k}\cdot\sqrt[k]{a_{k+1}a_{k+2}\cdot\ldots\cdot a_{2k}}} \le \frac{\sqrt[k]{a_1a_2\cdot\ldots\cdot a_k}+\sqrt[k]{a_{k+1}a_{k+2}\cdot\ldots\cdot a_{2k}}}{2} \le$$

$$\frac{1}{2}\left(\frac{a_1+a_2+\ldots+a_k}{k}+\frac{a_{k+1}+a_{k+2}+\ldots+a_{2k}}{k}\right)=\frac{a_1+a_2+\ldots+a_{2k}}{2k}.$$

6.17) The proof is almost word for word, except for trivial adjustments, as given in Example 6.8, following the scheme P(k) ⇒ P(2k) and then P(k) ⇒ P(k - 1).

6.18) The inductive step, if one follows a casual argument, amounts to the *false* inequality $\frac{(2k+1)^2}{(2k+2)^2} < \frac{3k}{3k+3}$. This is remedied if we prove the stronger statement $\frac{1^2}{2^2} \cdot \frac{3^2}{4^2} \cdot \ldots \cdot \frac{(2n-1)^2}{(2n)^2} < \frac{1}{3n+1}$, which leads to requiring the correct, and easily verifiable, inequality $\frac{(2k+1)^2}{(2k+2)^2} < \frac{3k+1}{3k+4}$.

6.19) The strengthened inequality $\frac{1}{2\sqrt{1}} + \frac{1}{3\sqrt{2}} + \ldots + \frac{1}{(n+1)\sqrt{n}} < 2 - \frac{2}{\sqrt{n+1}}$ is an easy induction exercise.

6.20) The strengthened inequality $(1+\frac{1}{2^3})(1+\frac{1}{3^3}) \cdot \ldots \cdot (1+\frac{1}{n^3}) \le 3 - \frac{1}{n}$ is an easy induction exercise.

7.1) The inductive hypothesis on n can be used to show that the right hand side is a product of integers.

7.2) For 0 ≤ x < 1/n the result is trivial. Assume now the equation is true for all x with k/n ≤ x < (k+1)/n and let y be an arbitrary number with (k+1)/n ≤ y < (k+2)/n. We are to prove the stated identity for y. For this apply the hypothesis to x = y – 1/n. We will obtain the required sum for y except for the end terms, but these can be adjusted using [a + 1] = [a] + 1. Negative x can be sorted similarly.

7.3) For the inductive step argue as follows: Given k+1 stations it is clear (by the original assumption) that there is station, call it A, whose gasoline will take the car to the next station moving, say, clockwise. If we carry the gasoline of station A to the next station, we will have k stations and so, by the inductive hypothesis, the circuit can be completed. Suppose this method of completing the circuit starts from station B and moves clockwise. It is now clear that if we return the gasoline back to station A, the circuit can also be completed clockwise starting from B, and collecting the gasoline from A as we pass along.

8.1) For the inductive step observe that if $n^2 + n < k \le (n + 1)^2 + (n + 1)$ then $n^2 + n + ¼ < k < (n + 1)^2 + (n + 1) + ¼$ (the first inequality is valid because k is an *integer* greater than the *integer* $n^2 + n$). Thus $(n + ½)^2 < k < (n + 1 + ½)^2$ and so $\{\sqrt{k}\} = n + 1$, from which it follows that the contribution of the new terms in the sum is $\sum_{k=n^2+n+1}^{(n+1)^2+(n+1)} \{\sqrt{k}\} = (n+1)[(n+1)^2 + (n+1) - (n^2+n)] = 2(n+1)^2,$ as needed.

8.2) The case n = 1 is clear. Assume that for n = k we need at least k + 1 lines to cover the lattice points described. Then for n = k + 1 we have a) if one of the straight lines is x + y = k + 1, then we need, by hypothesis, at least k + 1 lines to cover the rest of the points. Total: k + 2. b) If, on the other hand, the line x + y = k + 1 is not included, then the k + 2 lattice points on it require at least k + 2 other lines to be covered.

8.3) First note that if N ≥ 999 then N – 1 ≥ s(N) because N – s(N) = $\sum_{k=1}^{n} a_k(10^k - a_k) + a_0(1-a_0)$ ≥ $1.(10^2 - 9) + 9.(1 - 9) > 0$. Suppose now that repeated application of s on each k with k ≤ n (where n ≥ 999) eventually comes to 1 or the cycle mentioned. Then for n+1 we have n ≥ s(n + 1). Hence repeated application of s to s(n + 1) behaves as claimed.

8.4) The first few terms of the sequence are 1, 1, 1, 2, 3, 7, 11, 26, 41, ... One may suspect that this sequence also satisfies the recurrence $a_{n+2} = 4a_n - a_{n-2}$ ($n \geq 1$). This in itself is easy to verify inductively from the given relation: $a_{n+3} = (1 + a_{n+1}a_{n+2})/a_n =$

$[1 + a_{n+1}(4a_n - a_{n-2})]/a_n = 4a_{n+1} + (1 - a_{n+1}a_{n-2})/a_n = 4a_{n+1} + [1 - (1 + a_na_{n-1})]/a_n =$

$4a_{n+1} - a_{n-1}$. The fact now that (a_n) consists of integers is clear for the new recurrence.

8.5) Make m the subject of induction. For the inductive step, in $a_m = \dfrac{(mn)!}{m!(n!)^m}$, where n if fixed, we have

$$\frac{a_{m+1}}{a_m} = \frac{(mn+1)(mn+2)...(mn)...(mn+n)}{(m+1)n!} = \frac{(mn+1)(mn+2)...(mn)...(mn+n-1)}{(n-1)!}.$$

Note that the first part of the numerator is the product of n-1 consecutive terms, so it is a multiple of the denominator (this can be proved in a number of ways, including inductively).

8.6) The validity of the inductive step, from n = m to n = m + 1, amounts to showing that

$$a_{m+1}(b_1 + b_2 + ... + b_m) + b_{m+1}(a_1 + a_2 + ... + a_m) \leq (a_1b_1 + a_2b_2 + ... + a_mb_m) + (m+1)\, a_{m+1}\, a_{m+1}.$$

This follows by summing k from 1 to m the easily verified inequalities

$$a_{m+1}\, b_k + a_k\, b_{m+1} \leq a_{m+1}\, b_{m+1} + a_k\, b_k \quad (1 \leq k \leq m).$$

The corresponding inequality when the sequences (a_n), (b_n) are increasing simply the Chebychev inequality reversed. The proof of this is identical to the previous except for reversing the "≤".

8.7) Let $S_n = \{x_1, x_2, \ldots, x_n\}$ ($n \in \mathbf{N}$). An arrangement of the subsets of S_1 as required, is $\varnothing$, $\{x_1\}$, $\{x_1, x_2\}$, $\{x_2\}$. For the induction step, assume that the subsets of S_n are arranged as required, labelled as $\varnothing = A_1, A_2, \ldots, A_{2^n}$. The following labeling of the 2^{n+1} subsets of S_{n+1} is easily seen to be suitable:

$\varnothing = A_1, A_2, \ldots, A_{2^n}, \{x_{n+1}\} \cup A_{2^n}, \{x_{n+1}\} \cup A_{2^n-1}, \{x_{n+1}\} \cup A_{2^n-2}, \ldots, \{x_{n+1}\} \cup A_1$.

8.8) For the induction step, with 2n + 2 points, pick any two that are joined with a segment. If there is a third point joined to both these two, we are done. Otherwise, the remaining 2n points are joined to these two with at most 2n segments (because each of the other points is joined to at most one of the two). Thus, the remaining 2n points are joined with at least $(n + 1)^2 - 2n = n^2 + 1$ segments. The result follows by our hypothesis.

8.9) For the inductive step assume the conclusion true if n = k. Take now a subset A of {1, 2, … , 2k + 1, 2k + 2} with k + 2 elements. If $2k + 2 \notin A$ then (at least) k + 1 elements of A are in {1, 2, … , 2k} and the conclusion follows by the inductive hypothesis. If $2k + 2 \in A$ then either $k + 1 \in A$ or $k + 1 \notin A$. In the first case we are done, so assume the second. Consider the set B consisting of the elements of A but with 2k + 2 replaced by k + 1. Note that B has (at least) k + 1 elements in {1, 2, … , 2k} (because it has k + 2 elements in {1, 2, … , 2k + 1}). By the inductive hypothesis there exist x, y in B such that $x \mid y$. If $x \neq k + 1 \neq y$, we are done. Otherwise we have y = k + 1 (we cannot have x = k + 1 because then $2k + 2 \geq y \geq 2x = 2k + 2$, yet $2k + 2 \notin B$). But then $x \mid 2y$.

8.10) For n = 2 any two points at unit distance apart would do. If for n = k there is a finite set A_k as described. Let $\vec{u}$ be any unit vector of the plane, different from the (necessarily finite in number) vectors joining points of A_k. We define A_{k+1} as consisting of the points in A_k together with the same points shifted by $\vec{u}$. It is clear that every point of A_{k+1} is at distance 1 from k+1 points of A_{k+1} (k of which are the responsibility of A_k but there is a new one to, due to $\vec{u}$).

8.11) For n = 2 such a matrix can be trivially written down, with 1's down the diagonal. If one constructs a matrix (a_{ij}) as stated for n = k, with the additional property that it has 1's down the diagonal, we can construct a corresponding 2kx2k matrix (b_{ij}) as follows: For $0 \leq i, j \leq n$, set a) $b_{i,j} = a_{i,j}$, b) $b_{i+n,j+n} = a_{i,j}$; c) $b_{i,j+n} = a_{i,j} + 2n$ and, finally, d) $b_{i+n,i} = 2n$ and $b_{i+n,j} = a_{i,j} + 2n$ for i not equal j.

It is easy now to check that (b_{ij}) has the required properties (and 1's down the diagonal). Note that this procedure constructs matrices of dimensions $2^m \times 2^m$, all m.

INEQUALITIES

Mircea Becheanu, Vasile Berinde
ROMANIAN MATHEMATICAL SOCIETY, NORTH UNIVERSITY OF BAIA MARE

In the first section about inequalities we presented the first principles of the theory of inequalities and some introductory techniques in proving inequalities.

We will develop further these techniques in the present section. We begin with a presentation of the basic inequalities and then we give a set of problems and exercises with the aim of introducing the readers in a wonderful land of mathematics. Most techniques involved in solving the collection problems are applications and refinements of the methods used in proving classical basic inequalities. Sometimes, it is necessary to combine two or several of these methods.

1. Means Inequalities

We recall that, given positive numbers $a_1, a_2, ..., a_n$, the following functions are defined:

AM (arithmetic mean): $A_n = \frac{a_1 + a_2 + ... + a_n}{n}$

GM (geometric mean): $G_n = \sqrt[n]{a_1 a_2 ... a_n}$

HM (harmonic mean): $H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}$

QM (quadratic mean): $Q_n = \sqrt{\frac{a_1^2 + a_2^2 + ... + a_n^2}{n}}$.

In the previous Chapter devoted to inequalities (Inequalities, Level 1), we have proved the following inequalities:

$$H_2 \le G_2 \le A_2 \le Q_2,$$
$$H_3 \le G_3 \le A_3 \le Q_3.$$

The aim of this section is to show that the inequalities

$$H_n \le G_n \le A_n \le Q_n$$

also hold, for all natural numbers n, $n \ge 2$. The basic step to show this, is to prove the following theorem which is considered to be the main result:

1.1. Theorem. *For all* $n \ge 2$ *and any positive numbers* $a_1, a_2, ..., a_n$ *the following AM-GM inequality holds:*

$$\frac{a_1 + a_2 + ... + a_n}{n} \ge \sqrt[n]{a_1 a_2 ... a_n},$$

and equality occurs if and only if $a_1 = a_2 = ... = a_n$.

It is worth mentioning that this inequality has many proofs, based on different ideas. We will present here three of them.

First proof (by standard induction)

We have seen that the theorem is true for $n=2$ and $n=3$. Assume it is true for n arbitrary positive numbers and we show that it is true for any $n+1$ positive numbers $a_1, a_2, ..., a_n, a_{n+1}$.
Since the *AM* and *GM* have symmetric expressions in $a_1, a_2, ..., a_n$, one may assume that $0 < a_1 \le a_2 \le ... \le a_{n+1}$. Also, one may assume that $a_1 < a_{n+1}$. Then, the following inequalities hold: $a_1 < \dfrac{a_1 + ... + a_{n+1}}{n+1} < a_{n+1}$

Let us denote $A_{n+1} = \dfrac{a_1 + ... + a_{n+1}}{n+1}$. Then $(a_1 - A_{n+1})(A_{n+1} - a_{n+1}) > 0$ or, equivalently,

$$a_1 + a_{n+1} - A_{n+1} > \frac{a_1 a_{n+1}}{A_{n+1}}. \qquad (1)$$

Let us consider the following n numbers: $a_2, a_3, ... a_n$, $a_1 + a_{n+1} - A_{n+1}$. Their arithmetic mean is

$$A = \frac{a_2 + ... + a_n + (a_1 + a_{n+1} - A_{n+1})}{n} = A_{n+1} \qquad (2)$$

and their geometric mean is

$$G = \sqrt[n]{a_2 a_3 ... a_n (a_1 + a_{n+1} - A_{n+1})}.$$

By inequality (1), we have

$$G > \sqrt[n]{a_2 a_3 ... a_n \cdot \frac{a_1 a_{n+1}}{A_{n+1}}} = \sqrt[n]{\frac{G_{n+1}^{n+1}}{A_{n+1}}}.$$

By induction, $A > G$. Therefore $A^n > G^n$. Combining with the above inequality and (2) one obtains: $A_{n+1}^{n+1} > G_{n+1}^{n+1}$. This proves the required result.

It is clear by the proof that if the numbers $a_1, ..., a_n$ are not equal, then $a_1 < a_{n+1}$ and
the inequality is strict.

Second proof (by induction up and down).
First step is to prove by induction on p, where $p \ge 1$, that $A_{2^p} \ge G_{2^p}$. Indeed, assuming that $A_{2^{p-1}} \ge G_{2^{p-1}}$, one has

$$\frac{a_1 + ... + a_{2^p}}{2^p} = \frac{\dfrac{a_1 + ... + a_{2^{p-1}}}{2^{p-1}} + \dfrac{a_{2^{p-1}+1} + ... + a_{2^p}}{2^{p-1}}}{2} \ge$$
$$\ge \sqrt{\sqrt[2^{p-1}]{a_1 ... a_{2^{p-1}}} \sqrt[2^{p-1}]{a_{2^{p-1}+1} ... a_{2^p}}} = \sqrt[2^p]{a_1 a_2 ... a_{2^p}}.$$

Now, we will show that if $A_{n+1} \ge G_{n+1}$ then $A_n \ge G_n$.
Let $a_1, ..., a_n$ be positive numbers. We apply $A_{n+1} \ge G_{n+1}$ to the numbers $a_1, a_2, ..., a_n, A_n$. Therefore,

$$A_n = \frac{a_1 + ... + a_n + A_n}{n+1} \ge \sqrt[n+1]{a_1 ... a_n \frac{a_1 + ... + a_n}{n}} = \sqrt[n+1]{G_n^n \cdot A_n}.$$

Taking the $(n+1)^{th}$ power one obtains $A_n^n \ge G_n^n$, which proves the result.

To end, it is sufficient to remark that by performing an induction going-up one obtains the result for $2,4,8,\ldots,2^p,\ldots$ numbers and then by combining it with a going-down argument, we can cover all positive integers n

Third proof (Ehlers).

We prove by induction on n the following statement: if $x_1, x_2,\ldots,x_n$ are positive numbers such that $x_1x_2\ldots x_n = 1$ then

$$x_1 + x_2 + \ldots + x_n \geq n .$$

For $n = 2$, it is obvious. Assume the statement is true for n and let $x_1x_2\ldots x_nx_{n+1} = 1$, $x_i > 0$.

There are two numbers, say x_1, x_2 such that $x_1 \geq 1$ and $x_2 \leq 1$. Then $(x_1-1)(x_2-1) \leq 0$. It is convenient to write this inequality in the form

$$x_1 x_2+1 \leq x_1+x_2$$

Then, $(x_1 + x_2) + (x_3 + \ldots + x_{n+1}) \geq 1 + x_1x_2 + (x_3 + \ldots + x_{n+1}) \geq 1 + n = n + 1.$

Returning to the proof, now, it is sufficient to apply Ehlers argument to the numbers

$$x_1 = \frac{a_1}{G_n}, \frac{a_2}{G_n},\ldots, \frac{a_{n-1}}{G_n}, \frac{a_n}{G_n}$$

1.2. Corollary. *For all $n \geq 2$ and any positive numbers $a_1,a_2,\ldots,a_n$ the following GM-HM inequality holds:*

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} \leq \sqrt[n]{a_1a_2\ldots a_n} .$$

Proof. One applies *AM-GM* inequality to the numbers $\frac{1}{a_1}, \frac{1}{a_2},\ldots, \frac{1}{a_n}$, to obtain:

$$\frac{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}{n} \geq \sqrt[n]{\frac{1}{a_1a_2\ldots a_n}} = \frac{1}{\sqrt[n]{a_1a_2\ldots a_n}} .$$

1.3. Examples.

1. Let $a_1,a_2,\ldots,a_n$ be positive numbers and let $b_1,b_2,\ldots,b_n$ be a permutation of them. Then

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} \geq n .$$

Solution. By the *AM-GM* inequality applied to the numbers $\frac{a_1}{b_1}, \frac{a_2}{b_2},\ldots, \frac{a_n}{b_n}$ we have:

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} \geq n\sqrt[n]{\frac{a_1a_2\ldots a_n}{b_1b_2\ldots b_n}} = n.$$

2. Let a,b be positive real numbers and α,β be positive rational numbers such that $\alpha + \beta = 1$. Then

$$\alpha a + \beta b \geq a^\alpha \cdot b^\beta ,$$

and equality occurs if and only if $a = b$.

Solution. Assume that $\alpha=\dfrac{m}{p}$, $\beta=\dfrac{n}{p}$ where $m,n,p\in\mathbb{N}$ and $m+n=p$. Applying the *AM-GM* inequality to the $m+n$ numbers:

$$a_1=...=a_m=a;\qquad b_{m+1}=...=b_{m+n}=b,$$

we see that

$$\frac{ma+nb}{m+n}\ge\sqrt[m+n]{a^mb^n}.$$

It can be written as

$$\frac{m}{m+n}a+\frac{n}{m+n}b=\alpha a+\beta b\ge a^\alpha b^\beta.$$

It is now clear by *AM-GM* inequality that equality requires $a=b$.

Remark. The above result has also the following variational interpretation: if α,β are given and a,b are real variables so that $\alpha a+\beta b=c$ is a constant, then the product $a^\alpha b^\beta$ attains its maximum value c when $a=b=c$.

1.3.3. For all positive integers n, the following inequality holds:

$$\sqrt[n]{n}<1+\frac{2}{\sqrt{n}}.$$

Solution. We apply *AM-GM* inequality to the n numbers $1,1,...,\sqrt{n},\sqrt{n}$.
We have

$$\sqrt[n]{n}=\sqrt[n]{1\cdot 1...\sqrt{n}\cdot\sqrt{n}}\le\frac{n-2+2\sqrt{n}}{n}=1-\frac{2}{n}+\frac{2}{\sqrt{n}}<1+\frac{2}{\sqrt{n}}.$$

The following example has important applications.

1.3.4. For any positive integer n set

$$a_n=\left(1+\frac{1}{n}\right)^n \text{ and } b_n=\left(1+\frac{1}{n}\right)^{n+1}.$$

Then, the following inequalities hold:

$$a_n<a_{n+1}<b_{n+1}<b_n.$$

Solution. For positive numbers a,b and arbitrary n, by example 1.3.2 one has

$$\frac{a+nb}{n+1}\ge\left(ab^n\right)^{\frac{1}{n+1}},$$

and equality occurs if and only if $a=b$. Hence,

$$\left(\frac{a+nb}{n+1}\right)^{n+1}\ge ab^n.$$

Taking $a=1$ and $b=1+\dfrac{1}{n}$ in the above inequality, we obtain

$$\left(\frac{n+2}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^n,$$

which shows that $a_{n+1}>a_n$.

Taking $a=\dfrac{k+2}{k}$, $b=\dfrac{k+2}{k+1}=1+\dfrac{1}{k+1}$ and $n=k+1$, we obtain

$$\left(\frac{\frac{k+2}{k}+(k+2)}{k+2}\right)^{k+2} > \frac{k+2}{k}\left(1+\frac{1}{k+1}\right)^{k+1}.$$

Therefore,

$$\left(\frac{k+1}{k}\right)^{k+2} > \frac{k+2}{k}\left(1+\frac{1}{k+1}\right)^{k+1}.$$

Since $\frac{k+2}{k+1}=1+\frac{1}{k+1}$ we obtain

$$\frac{k+1}{k}\left(1+\frac{1}{k}\right)^{k+1} > \frac{k+1}{k}\left(1+\frac{1}{k+1}\right)^{k+2}.$$

This shows that $b_k > b_{k+1}$, which completes the proof.

1.4. Theorem. *For all $n \geq 2$ and any positive numbers $a_1, a_2, ..., a_n$ the following AM-QM inequality holds:*

$$\frac{a_1+a_2+...+a_n}{n} \leq \sqrt{\frac{a_1^2+a_2^2+...+a_n^2}{n}}.$$

The equality holds if and only if $a_1 = a_2 = ... = a_n$.

Proof. By squaring the inequality one obtains the equivalent form:

$$\left(a_1+a_2+...+a_n\right)^2 \leq n\left(a_1^2+a_2^2+...+a_n^2\right).$$

There are several methods to prove this.

First method uses Cauchy-Schwarz inequality in the form (2.2.2, Inequalities, Level 1), see also section 2 in the present Chapter:

$$a_1^2+a_2^2+...+a_n^2 = \frac{a_1^2}{1}+\frac{a_2^2}{1}+...+\frac{a_n^2}{1} \geq \frac{(a_1+a_2+...+a_n)^2}{n}.$$

Equality holds if and only if $a_1 = a_2 = ... = a_n$.

Second method is a direct proof. The left hand side of inequality is:

$$\left(a_1+a_2+...+a_n\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \quad .$$

Then the required inequality becomes:

$$2\sum_{i<j} a_i a_j \leq (n-1)\left(a_1^2+a_2^2+...+a_n^2\right).$$

(3)

The sum in the left hand side contains $\frac{n(n-1)}{2}$ terms. For each pair (i,j) with $1 \leq i < j \leq n$ we have:

$$a_i a_j \leq \frac{a_i^2+a_j^2}{2}.$$

By adding all these inequalities one has

$$\sum_{i<j} a_i a_j \leq \frac{1}{2}\sum_{i<j}\left(a_i^2+a_j^2\right).$$

In the right hand side of the above inequality each a_i^2 appears $(n-1)$ times since it appears with all $a_j^2,\ j \neq i$. Hence,

$$\sum_{i<j}\left(a_i^2 + a_j^2\right) = (n-1)\sum_{i=1}^{n} a_i^2 \ .$$

which ends the proof.

2. Cauchy-Schwarz inequality

2.1. Proofs of Cauchy-Schwarz inequality

This inequality was already stated in (2.2.2) (Inequalities, Level 1). For all real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ the following Cauchy-Schwarz inequality holds

$$\left(a_1 b_1 + ... + a_n b_n\right)^2 \leq \left(a_1^2 + ... + a_n^2\right)\left(b_1^2 + ... + b_n^2\right).$$

The equality occurs if and only if the numbers $a_1, ..., a_n$ and $b_1, ..., b_n$ are proportional, that is

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n} .$$

In these ratios, if $b_i = 0$ then $a_i = 0$.
There are several different proofs of this Cauchy-Schwarz inequality.

First proof (Quadratic functions).
Consider the quadratic function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = (a_1 x - b_1)^2 + (a_2 x - b_2)^2 + ... + (a_n x - b_n)^2 .$$

It is clear that $f(x) \geq 0$, for all $x \in \mathbb{R}$, and $f(x) = 0$ if and only if there exists $\lambda \in \mathbb{R}$ such that $b_i = \lambda a_i$, for all $i = 1, 2, ..., n$. Writing $f(x)$ under the canonical form of a quadratic function we have:

$$f(x) = \left(\sum_{i=1}^{n} a_i^2\right) x^2 - 2\left(\sum_{i=1}^{n} a_i b_i\right) x + \sum_{i=1}^{n} b_i^2 .$$

Its discriminant is

$$\Delta = 4\left(\sum_{i=1}^{n} a_i b_i\right)^2 - 4\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 .$$

Since $\sum_{i=1}^{n} a_i^2 > 0$ it follows that $\Delta \leq 0$, and the result follows.
The case of equality follows by the
Second proof (based on Lagrange's identity).

It is clear that we can compute the difference between right hand side and left hand side to obtain the equality:

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{i<j}\left(a_i b_j - a_j b_i\right)^2$$

(It is called Lagrange's identity). The proof of this identity is very elementary and it requires only a careful computation of all members in both sides of equality. The product $\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$ contains terms of the form $a_i^2 b_i^2$ for $i = 1,2,...,n$ and terms of the form $a_i^2 b_i^2$ for all $i \neq j$. Each monomial $a_i^2 b_i^2$ appears twice.

The square $\left(\sum_{i=1}^{n} a_i b_i\right)^2$ can be computed as follows:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{i=1}^{n} a_i^2 b_i^2 + 2\sum_{i<j} a_i b_i a_j b_j \ .$$

We cancel the terms $a_i^2 b_i^2$ and in left hand side remain exactly the monomials which can be condensed in

$$\sum_{i<j} (a_i b_j - a_j b_i)^2 \ .$$

2.2. Applications

2.2.1. For any positive numbers $a_1, a_2, ..., a_n$, the following inequality holds:

$$(a_1 + ... + a_n)\left(\frac{1}{a_1} + ... + \frac{1}{a_n}\right) \geq n^2 .$$

Solution. Apply Cauchy-Schwarz inequality to the numbers $\sqrt{a_1},...,\sqrt{a_n}$, $\frac{1}{\sqrt{a_1}},...,\frac{1}{\sqrt{a_n}}$.

Remark. This inequality can also be obtained from *AM-HM* inequality and from AM-GM as well.

2.2.2. For all real numbers $a_1, a_2, ..., a_n$ the following inequality holds

$$(a_1 + ... + a_n)^2 \leq n\left(a_1^2 + ... + a_n^2\right).$$

This result was already proved in Theorem 1.4. A new proof can be obtained by using Cauchy-Schwarz applied to the numbers $a_1, ..., a_n$ and $1, ..., 1$.

3. Rearrangement inequality and Tchebyshev's inequality

3.1. Rearrangement inequality

Let $a_1 \geq a_2 \geq ... \geq a_n$ and $b_1 \geq b_2 \geq ... \geq b_n$ be sequences of real numbers. Then the following inequality holds:

$$a_1 b_1 + a_2 b_2 + ... + a_n b_n \geq a_1 b_n + a_2 b_{n-1} + ... + a_n b_1 . \quad (4)$$

Indeed, when subtracting the right hand side from the left hand side one obtains the equivalent form:

$$a_1(b_1 - b_n) + a_2(b_2 - b_{n-1}) + ... + a_n(b_n - b_1) \geq 0 .$$

It can be written again as:

$$(a_1 - a_n)(b_1 - b_n) + (a_2 - a_{n-1})(b_2 - b_{n-1}) + ... + (a_k - a_{k+i})(b_k - b_{k+i}) \geq 0 .$$

where $k = \left[\frac{n}{2}\right]$ and

$$i = \begin{cases} 1 \text{ when } n \text{ is even} \\ 2 \text{ when } n \text{ is odd}. \end{cases}$$

The last inequality is obvious.

The inequality (4) can be refined in the following way: if $b_{i_1}, b_{i_2}, ..., b_{i_n}$ is a permutation of the numbers $b_1, b_2, ..., b_n$, then the double inequality holds

$$\sum_{i=1}^{n} a_i b_i \ge a_1 b_{i_1} + a_2 b_{i_2} + ... + a_n b_{i_n} \ge \sum_{i=1}^{n} a_i b_{n+1-i} . \qquad (5)$$

In other words, the sum $\sum_{k=1}^{n} a_k b_{i_k}$ attains its maximum value when the numbers $b_1, b_2, ..., b_n$ are considered is descending order and attains its minimum value when they are considered in ascending order. For this reason, the inequality (5) is called the rearrangement inequality (RI). We shall see how the inequality

$$\sum_{i=1}^{n} a_i b_i \ge \sum_{k=1}^{n} a_k b_{i_k}$$

can be proved.

The idea is to show that the permutation $b_{i_1}, ..., b_{i_n}$ for which $\sum_{k=1}^{n} a_k b_{i_k}$ is a maximum is the identity permutation $b_1, b_2, ..., b_n$. Assume that $i_1 \ne 1$ and let i_k be the index for which $b_{i_k} = b_1$. Consider the permutation $b_{i_k} = b_1, b_{i_2}, ..., b_{i_{k-1}}, b_{i_{k+1}}, ..., b_{i_n}$ and compute the difference:

$$\sum_{k=1}^{n} a_k b_{i_k} - (a_1 b_1 + a_2 b_{i_2} + ... + a_k b_{i_1} + ... + a_n b_{i_n}) =$$
$$= a_1(b_{i_1} - b_1) + a_k(b_{i_k} - b_{i_1}) = (a_1 - a_k)(b_1 - b_{i_1}) \le 0 .$$

So, if $\sum_{k=1}^{n} a_k b_{i_k}$ is a maximum, then $i_1 = 1$.

The procedure can be applied again to show that $i_2 = 2, ..., i_n = n$.

3.2. Tchebyshev's inequality

As an application of the inequality (5) we can prove a very interesting and powerful inequality due to Tchebyshev: if $a_1 \ge a_2 \ge ... \ge a_n$ and $b_1 \ge b_2 \ge ... \ge b_n$ then

$$(a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n) \le n(b_1 + a_2 b_2 + ... + a_n b_n). \qquad (6)$$

Sometimes, this inequality is given under a form which involves arithmetical means and which makes it more adequate and attractive:

$$\frac{a_1 + ... + a_n}{n} \cdot \frac{b_1 + ... + b_n}{n} \le \frac{a_1 b_1 + ... + a_n b_n}{n} .$$

We give one of several methods to prove (6). The left hand side is a sum of n^2 products

$$\sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j = \sum_{i,j=1}^{n} a_i b_j$$

which can be arranged in sums of the form $\sum_{k=1}^{n} a_k b_{i_k}$.

For instance, we can organize them such that the numbers $b_1, b_2, ..., b_n$ are permuted cyclically into $b_2, b_3, ..., b_n, b_1$, then $b_3, b_n, ..., b_2$, an so on. Therefore

$$\sum_{i,j=1}^{n} a_i b_j = (a_1b_1 + a_2b_2 + ... + a_nb_n) + (a_1b_2 + a_2b_3 + ... + a_nb_1) + ...$$
$$... + (a_1b_n + a_2b_1 + ... + a_nb_{n-1})$$

Each bracket is dominated by $a_1b_1 + a_2b_2 + ... + a_nb_n$ and by adding them, one obtains (6).

3.3. Applications

3.3.1. The *AM-QM* inequality (from Theorem 1.4) is a particular case of Tchebyshev's inequality:

$$(a_1 + a_2 + ... + a_n)^2 \le n\,(a_1^2 + a_2^2 + ... + a_n^2).$$

Indeed, since the inequality is symmetric, one may assume that $a_1 \ge a_2 \ge ... \ge a_n$. Take also the numbers $b_1 = a_1, b_2 = a_2, ..., b_n = a_n$. It is clear that Tchebyshev's inequality applied to this case yields the required result.

3.3.2. For any positive real numbers a, b, c the following inequality holds:

$$\frac{a^2 + b^2}{2c} + \frac{b^2 + c^2}{2a} + \frac{c^2 + a^2}{2b} \ge a + b + c.$$

Solution. Assume that $a \ge b \ge c > 0$. It follows that $\frac{1}{c} \ge \frac{1}{b} \ge \frac{1}{a} > 0$. By applying twice the rearrangement inequality we have:

$$a^2 \cdot \frac{1}{a} + b^2 \cdot \frac{1}{b} + c^2 \cdot \frac{1}{c} \le \frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b}$$

and

$$a^2 \cdot \frac{1}{a} + b^2 \cdot \frac{1}{b} + c^2 \cdot \frac{1}{c} \le \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

By adding these inequalities one has the required result.

3.3.3. In any acute triangle *ABC*, the following inequalities hold:

$$2(a\sin A + b\sin B + c\sin C) \ge$$
$$\ge a(\sin B + \sin C) + b(\sin C + \sin A) + c(\sin A + \sin B); \qquad \text{(a)}$$
$$3(a\sin A + b\sin B + c\sin C) \ge (a + b + c)(\sin A + \sin B + \sin C). \qquad \text{(b)}$$

Proof. Assume $a \ge b \ge c$. Then $\sin A \ge \sin B \ge \sin C$. Apply successively the rearrangement inequality to the triples (a, b, c) and $(\sin A, \sin B, \sin C)$, the last numbers being arranged as $(\sin A, \sin B, \sin C)$. Summing up the obtained inequalities one has (a). To obtain (b), we add $a\sin A + b\sin B + c\sin C$ to both members of (a).

3.3.4. For any positive numbers a, b, c, the following inequality holds:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

This was mentioned and proved in (Inequalities, Level 1).
A new proof can be given by using Tchebyshev's argument for ordered triples $a \geq b \geq c$ and $\frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b}$.

We have:

$$3\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\geq(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)=$$
$$=\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}=3+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

The result is now obvious.

3.3.5. Let $x_1, x_2, ..., x_n$ and α, β be positive real numbers such that $\alpha < \beta$ and $x_1 x_2 ... x_n = 1$. Then

$$x_1^{\alpha}+x_2^{\alpha}+...+x_n^{\alpha}\leq x_1^{\beta}+x_2^{\beta}+...+x_n^{\beta}.$$

Proof.

We use Tchebyshev's inequality. Assume that $x_1 \geq x_2 \geq ... \geq x_n$ and consider the ordered systems

$$x_1^{\alpha}\geq x_2^{\alpha}\geq...\geq x_n^{\alpha}, x_1^{\beta-\alpha}\geq x_2^{\beta-\alpha}\geq...\geq x_n^{\beta-\alpha}$$

By Tchebyshev's inequality one has:

$$n\left(x_1^{\beta}+x_2^{\beta}+...+x_n^{\beta}\right)\geq$$
$$\geq\left(x_1^{\alpha}+x_2^{\alpha}+...+x_n^{\alpha}\right)\left(x_1^{\beta-\alpha}+x_2^{\beta-\alpha}+...+x_n^{\beta-\alpha}\right).$$

By *AM-GM* inequality

$$x_1^{\beta-\alpha}x_2^{\beta-\alpha}+...+x_n^{\beta-\alpha}\geq n\sqrt[n]{(x_1x_2...x_n)^{\beta-\alpha}}=n.$$

1. A new proof of the *AM-GM* inequality:

$$\sqrt[n]{(x_1x_2...x_n)}\leq\frac{x_1+x_2+...+x_n}{n},$$

for all positive numbers $x_1, x_2, ..., x_n$.

Denote $G=\sqrt[n]{x_1x_2...x_n}$ and consider the systems of numbers:

$$a_1=\frac{x_1}{G},\ a_2=\frac{x_1x_2}{G^2},...,a_n=\frac{x_1x_2...x_n}{G^n}=1 \text{ and}$$
$$b_1=\frac{1}{a_1},\ b_2=\frac{1}{a_2},...,b_n=\frac{1}{a_n}.$$

The ordering of these numbers is opposite, so by (4) one has:

$$a_1b_1+a_2b_2+...+a_nb_n\leq a_1b_n+a_2b_1+...+a_nb_{n-1}.$$

By replacing the expressions of a_i, b_i one has:

$$n \le \frac{x_1}{G} + \frac{x_1 x_2}{G^2} \cdot \frac{G}{x_1} + \frac{x_1 x_2 x_3}{G^3} \cdot \frac{G^2}{x_1 x_2} + \ldots + \frac{x_1 x_2 \ldots x_n}{G^n} \cdot \frac{G^{n-1}}{x_1 x_2 \ldots x_{n-1}}.$$

This yields

$$n \le \frac{x_1}{G} + \frac{x_2}{G} + \ldots + \frac{x_n}{G},$$

which is precisely the *AM-GM* inequality.

4. Bernoulli's inequality

4.1. Basic Bernoulli's inequality

A lot of inequalities are known as being due to Bernoulli. The basic Bernoulli's inequality is the following: if $x > -1$ is a real number, and n is a positive integer, then

$$(1+x)^n \ge 1 + nx. \tag{7}$$

It can be proved by induction on n. For $n = 1$, we have equality. Assume that $(1+x)^n \ge 1 + nx$ and multiply this by the positive number $1 + x$. We have:

$$(1+x)^{n+1} \ge (1+nx)(1+x) = 1 + (n+1)x + nx^2 \ge 1 + (n+1)x.$$

Hence, the inequality holds for all *n*.

4.2 Refined Bernoulli's inequality

The inequality (7) can be refined to several distinct numbers. Suppose that $n \ge 2$ and $x_1, x_2, \ldots, x_n$ are non-zero real numbers which have all the same sign and $x_i \ge -2$, for all
$i = 1, 2, \ldots, n$ Then

$$(1+x_1)(1+x_2)\ldots(1+x_n) > 1 + (x_1 + x_2 + \ldots + x_n). \tag{8}$$

For the proof of (8) we will use induction once again. For $n = 2$ one has:

$$(1+x_1)(1+x_2) = 1 + (x_1 + x_2) + x_1 x_2 > 1 + (x_1 + x_2).$$

If $x_1, \ldots, x_n$ are all positive, by standard computations which use symmetric polynomials we have

$$(1+x_1)(1+x_2)\ldots(1+x_n) = 1 + \sum_{i=1}^{n} x_i + \sum_{i<j} x_i x_j + \ldots$$
$$+ \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k} + \ldots + x_1 x_2 \ldots x_n < 1 + \sum_{i=1}^{n} x_i$$

Therefore, the case when $-2 < x_i < 0$ remains. Assume the inequality holds for n numbers and we want to prove it for $n+1$. We have

$$\begin{aligned}&(1+x_1)\ldots(1+x_n)(1+x_{n+1}) - (1 + x_1 + \ldots + x_n + x_{n+1}) = \\ &= \left[(1+x_1)\ldots(1+x_n) - (1 + x_1 + \ldots + x_n)\right] + \\ &+ x_{n+1}\left[(1+x_1)\ldots(1+x_n) - 1\right].\end{aligned}$$

Since $-2 \le x_k < 0$, it follows that $-1 \le 1 + x_k < 1$. So, in all cases $(1+x_1)\ldots(1+x_n) < 1$ and

$$x_{n+1}\left[(1+x_1)\ldots(1+x_n) - 1\right] > 0.$$

4.3 Application. If $a_1, a_2, \ldots, a_n$ are positive numbers not greater than 1, then

$$(a_1^2 + n - 1)(a_2^2 + n - 1)\ldots(a_n^2 + n - 1) \ge n^{n-1}(a_1 + a_2 + \ldots + a_n)^2.$$

The inequality is equivalent to

$$\left(\frac{a_1^2-1}{n}+1\right)\left(\frac{a_2^2-1}{n}+1\right)\ldots\left(\frac{a_n^2-1}{n}+1\right)\geq\frac{1}{n}(a_1+a_2+\ldots+a_n)^2.$$

By Bernoulli's inequality we have

$$\left(\frac{a_1^2-1}{n}+1\right)\left(\frac{a_2^2-1}{n}+1\right)\ldots\left(\frac{a_n^2-1}{n}+1\right)\geq 1+\sum_{I=1}^{N}\frac{a_i^2-1}{n}$$

and due to inequality 2.2.2, the conclusion follows.

5. Proposed problems

In this section we present some inequalities under the form of proposed problems. Solving them requires that the reader has a good knowledge of the basic inequalities explained previously.

5.1. An IMO problem with an unexpected solution[1]

Let n be a fixed integer with $n\geq 2$.
(a) Find the least constant C such that the inequality

$$\sum_{1\leq i<j\leq n}x_ix_j\left(x_1^2+x_j^2\right)\leq C\left(\sum_{i=1}^{n}x_i\right)^4$$

holds for all real numbers $x_1,\ x_2,\ldots,x_n\geq 0$.
(b) For this constant C, determine when equality holds.

First solution. For $n=2$, we have to find the maximum of the function

$$f(x,y)=\frac{xy(x^2+y^2)}{(x+y)^4}$$

when $x\geq 0,\ y\geq 0$ and x,y are not simultaneously zero.

Since the fraction defining f is homogeneous of degree 0, we denote $a=\dfrac{x}{x+y},\ b=\dfrac{y}{x+y}$, such that $a+b=1$. One obtains the function

$$g(a,b)=ab(a^2+b^2),$$

where $a+b=1$. After denoting $2ab=p$, one obtains a quadratic function in one variable

$$g(a,b)=ab(1-2ab)=\frac{1}{2}p(1-p)\leq\frac{1}{8}.$$

The inequality $\dfrac{1}{2}p(1-p)\leq\dfrac{1}{8}$ is obvious and the maximum $C=\dfrac{1}{8}$ is obtained for $p=\dfrac{1}{2}$, which means that $a=b=\dfrac{1}{2}$.

In the general case, we will prove that

$$\sum_{1\leq i<j\leq n}x_ix_j\left(x_i^2+x_j^2\right)\leq\frac{1}{8}\left(\sum_{i=1}^{n}x_i\right)^4.$$

[1] Problem 2 in IMO, 1999, Bucharest

Denote $\sum_{i=1}^{n} x_i = S$ and $y_i = \frac{x_i}{S}$. As above, we consider an auxiliary function f and prove that

$$f(y_1,...,y_n)=\sum_{i<j} y_i y_j \left(y_i^2+y_j^2\right) \leq \frac{1}{8}$$

for all $y_1,..., y_n \geq 0$ and $y_1 + ... + y_n = 1$.

Since the polynomial $f(y_1,..., y_n)$ is symmetric we may assume that $y_1 \geq y_2 \geq ... \geq 0$. The idea is to show that if $y_n > 0$ then,

$$f(y_1,..., y_{n-2}, y_{n-1} + y_n, 0) > f(y_1,..., y_{n-1}, y_n)$$

which requires only careful computations.

By repeating the above argument it follows that a maximum is attained when $n-2$ variables are zero, so the problem reduces to the case $n = 2$.

Second solution (due to M. Rădulescu)

For $n = 2$, as in the previous solution, we have

$$xy(x^2 + y^2) \leq \frac{1}{8}(x + y)^4.$$

Let us now turn to $n > 2$ numbers. Denote $M = \sum_{i=1}^{n} x_i^2$. Then $x_i^2 + x_j^2 \leq M$ and equality holds if and only if $x_k = 0$ for all k, $k \neq i, j$. By applying appropriately the *AM-GM* inequality one obtains:

$$\sum_{i<j} x_i x_j \left(x_i^2 + x_j^2\right) \leq \sum_{i<j} x_i x_j M = \frac{1}{2} M \sum_{i<j} 2 x_i x_j \leq$$

$$\leq \frac{1}{2}\left(\frac{M + 2\sum x_i x_j}{2}\right) = \frac{1}{8}\left(\sum_{i=1}^{n} x_i\right)^4.$$

The equality holds if and only if there exists a pair (i, j) such that $x_i = x_j$ and $x_k = 0$, for $k \neq i, j$.

5.2. An interesting application of Cauchy-Schwarz inequality[2]

Let n be a positive integer and let $x_1 \leq x_2 \leq ... \leq x_n$ *be real numbers.*

(a) Prove that

$$\left(\sum_{i,j=1}^{n} \left|x_i - x_j\right|\right)^2 \leq \frac{2(n^2 - 1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2 ;$$

(b) Show that equality holds if and only if $x_1, x_2,..., x_n$ *is an arithmetic progression.*

Solution. (a) Since both sides of the inequality remain invariant under a translation on the real axis, we may assume that $\sum_{i=1}^{n} x_i = 0$.

Then,

$$\sum_{i,j=1}^{n} \left|x_i - x_j\right| = 2\sum_{i<j} (x_j - x_i) = 2\sum_{i=1}^{n} (2i - n - 1) x_i .$$

[2] Problem 5 in IMO 2003, Tokyo

By the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i,j=1}^{n}\left|x_i - x_j\right|\right)^2 \le 4\sum_{i=1}^{n}(2i-n-1)^2\sum_{i=1}^{n}x_i^2 = \frac{4n(n+1)(n-1)}{3}\sum_{i=1}^{n}x_i^2 .$$

On the other hand

$$\sum_{i,j=1}^{n}\left(x_i - x_j\right)^2 = n\sum_{i=1}^{n}x_i^2 - \sum_{i=1}^{n}x_i\sum_{j=1}^{n}x_j + n\sum_{j=1}^{n}x_j^2 = 2n\sum_{i=1}^{n}x_i^2 .$$

Therefore, the required inequality yields.

(b) If the equality holds, then $x_i = k(2i - n - 1)$ for some k, which means that $x_1,...,x_n$ is an arithmetic progression.

Finally, it is easy to see that if $x_1,\ldots,x_n$ is an arithmetic progression, the inequality becomes equality.

5.3. *For all* $a_1,\ a_2,\ldots,\ a_n > 0$ *and* $S = a_1 + a_2 + \ldots + a_n$*, we have*

$$\frac{a_1}{S-a_1} + \frac{a_2}{S-a_2} + \ldots + \frac{a_n}{S-a_n} \ge \frac{n}{n-1}.$$

Solution 1. We use linear change of variables.

Solution 2. Since the left hand size is a symmetric function in $a_1,a_2,\ldots,a_n$, we may assume $a_1 \ge a_2 \ge \ldots \ge a_n$, so that $S-a_1 \le S-a_2 \le \ldots \le S-a_n$ and therefore

$$\frac{1}{S-a_1} \ge \frac{1}{S-a_2} \ge \ldots \ge \frac{1}{S-a_n} .$$

By Chebyshev's inequality we get

$$(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{S-a_1} + \frac{1}{S-a_2} + \ldots + \frac{1}{S-a_n}\right) \le n\left(\frac{a_1}{S-a_1} + \frac{a_2}{S-a_2} + \ldots + \frac{a_n}{S-a_n}\right)$$

(5.1)

By using the inequality from 2.2.1 we have

$$(S-a_1 + S-a_2 + \ldots + S-a_n)\left(\frac{1}{S-a_1} + \frac{1}{S-a_2} + \ldots + \frac{1}{S-a_n}\right) \ge n^2$$

which gives

$$\frac{1}{S-a_1} + \frac{1}{S-a_2} + \ldots + \frac{1}{S-a_n} \ge \frac{n^2}{(n-1)S}$$

The last inequality, together with (5.1) prove the required result.

5.4. *If* $a_1,\ a_2,\ldots,a_n > 0$ and $a_1 + a_2 + \ldots + a_n = 1$, *then*

$$\frac{a_1}{2-a_1} + \frac{a_2}{2-a_2} + \ldots + \frac{a_n}{2-a_n} \ge \frac{n}{2n-1}.$$

(Balkan Mathematical Olympiad).

Solution. Similar to 5.3, but using the numbers $2S - a_i = 2 - a_i$ instead of $S - a_i$, $i = 1,2,\ldots,n$.

5.5. *If* $a,b,c > 0$*, then*

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \ge \frac{a+b+c}{3}.$$

Solution. Denote

$$A=\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2},$$

$$A=\frac{b^3}{a^2+ab+b^2}+\frac{c^3}{b^2+bc+c^2}+\frac{a^3}{c^2+ca+a^2}.$$

Since

$$\frac{a^3-b^3}{a^2+ab+b^2}=a-b$$

it follows that $A=B$. Using the fact that $A=\dfrac{A+B}{2}$ and applying inequalities of the form

$$a^2+ab+b^2\le 3(a^2-ab+b^2),$$

equivalent to $(a-b)^2\ge 0$, one obtains the required inequality.

5.6. *If* $n\ge 3$ *and* $a_1,a_2,\ldots,a_n>0$, *then*

$$\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+\ldots+\frac{a_n^2}{a_n+a_1}\ge\frac{1}{2}(a_1+a_2+\ldots+a_n).$$

Solution. Denote

$$A=\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+\ldots+\frac{a_n^2}{a_n+a_1},$$

and

$$B=\frac{a_2^2}{a_1+a_2}+\frac{a_3^2}{a_2+a_3}+\ldots+\frac{a_1^2}{a_n+a_1}$$

and, similarly to the previous problem, show that $A=B$. Then use the same idea together with the inequalities

$$2(a_i^2+a_j^2)\ge(a_i+a_j)^2,\ i=1,2,\ldots,n.$$

Another solution can be obtained by using Cauchy-Schwarz inequality, revisited under the form

$$\sum_{i=1}^{n}\frac{a_i^2}{b_i}\le\frac{\sum a_i^2}{\sum b_i}\quad .$$

Indeed, by using cyclic summation, one has

$$\sum\frac{a_1^2}{a_1+a_2}\ge\frac{(\sum a_i)^2}{2\sum a_i}=\frac{1}{2}\sum a_i.$$

5.7. *If* $n\ge 3$ *and* $a_1,\ a_2,\ldots,a_n>0$, *then*

$$\frac{a_1^3}{a_1+a_2}+\frac{a_2^3}{a_2+a_3}+\ldots+\frac{a_n^3}{a_n+a_1}\ge\frac{1}{2}(a_1^2+a_2^2+\ldots+a_n^2).$$

By the Cauchy-Schwarz revisited inequality we have $(a_{n+1}=a_1)$

$$\sum_1^n \frac{a_i^3}{a_i + a_{i+1}} = \sum_1^n \frac{a_i^4}{a_i^2 + a_i a_{i+1}} \geq \frac{\left(\sum a_i^2\right)^2}{\sum a_i^2 + \sum a_i a_{i+1}}.$$

So, we have to prove the following:

$$\sum_1^n a_i^2 \geq \sum_1^n a_i a_{i+1},$$

which is exactly the Cauchy-Schwarz inequality:

$$a_1a_2 + a_2a_3 + \ldots + a_na_1 \leq \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2} \cdot \sqrt{a_2^2 + a_3^2 + \ldots + a_1^2}.$$

5.8. *If* $a, b, c > 0$ *and* $a^2 + b^2 + c^2 = 1$ *then*

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \geq \frac{3\sqrt{3}}{2}.$$

Solution. It is clear that $a, b, c \in (0,1)$. First of all we prove that for any $x > 0$ the inequality $x(1-x^2) \leq \frac{2}{3\sqrt{3}}$ holds. This is obvious if we rewrite it equivalently in the form $(x\sqrt{3}-1)^2(x\sqrt{3}+2) \geq 0$. The equality holds if and only if $x = \frac{1}{\sqrt{3}}$. For $x \in (0,1)$ the previous inequality is equivalent to $\frac{x}{1-x^2} \geq \frac{3\sqrt{2}}{2}x^2$. Now, take $x = a$, $x = b$, $x = b$ in the last inequality, sum up these three inequalities to obtain the inequality in our problem.

5.9. *Let* a, b, A, B *be positive numbers such that* $a < A$ *and* $b < B$*. For all numbers* $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ *such that* $a_i \in [a, A]$, $b_i \in [b, B]$, $\forall i = 1, 2, \ldots, n$*, the following inequality holds:*

$$\frac{(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2)}{(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2} \leq \frac{1}{4}\left(\sqrt{\frac{AB}{ab}} + \sqrt{\frac{ab}{AB}}\right)^2.$$

(This inequality is due to G. Polyá and G. Szegö)

Solution. From $a_i \in [a, A]$, $b_i \in [b, B]$ we obtain that

$$\frac{b}{A} \leq \frac{b_i}{a_i} \leq \frac{B}{a}, \quad i = 1, 2, \ldots, n$$

which gives

$$\left(\frac{b_i}{a_i} - \frac{b}{A}\right)\left(\frac{B}{a} - \frac{b_i}{a_i}\right) \geq 0, \quad i = 1, 2, \ldots, n.$$

This last inequality is equivalent to

$$b_i^2 + \frac{bB}{aA}a_i^2 \leq \left(\frac{B}{a} + \frac{b}{A}\right)a_ib_i, \quad i = 1, 2, \ldots, n.$$

By summing up these n inequalities, we obtain

$$\sum_{i=1}^n b_i^2 + \frac{bB}{aA}\sum_{i=1}^n a_i^2 \leq \left(\frac{B}{a} + \frac{b}{A}\right)\sum_{i=1}^n a_ib_i.$$

But
$$2\sqrt{\frac{bB}{aA}\sum_{i=1}^{n}b_i^2\sum_{i=1}^{n}a_i^2} \le \sum_{i=1}^{n}b_i^2 + \frac{bB}{aA}\sum_{i=1}^{n}a_i^2$$
and hence
$$2\sqrt{\frac{bB}{aA}\sum_{i=1}^{n}b_i^2\sum_{i=1}^{n}a_i^2} \le \left(\frac{B}{a}+\frac{b}{A}\right)\sum_{i=1}^{n}a_ib_i\ ,$$
which yields the desired inequality.

5.10. *For the permutation* $(i_1,i_2,\dots,i_n)$ *of the numbers* $1,2,\dots,n$ *and all positive numbers* $a_1,a_2,\dots,a_n$*, the following inequality holds:*
$$\left(1+\frac{a_1}{a_{i_1}}\right)\left(1+\frac{a_2}{a_{i_2}}\right)\cdot\ldots\cdot\left(1+\frac{a_n}{a_{i_n}}\right) \le \left(a_1+\frac{1}{a_1}\right)\left(a_2+\frac{1}{a_2}\right)\cdot\ldots\cdot\left(a_n+\frac{1}{a_n}\right).$$
(A problem by Dan Seclăman, Gazeta Matematică, Bucharest)

Solution. Using the inequalities $a_i+\dfrac{1}{a_i}\ge 2,\ i=1,2,\dots,n$ we deduce that the right hand side is at least 2^n. To prove the inequality in the problem, it suffices to show that the left hand side is at most 2^n. To this end, apply first $AM-GM$ inequality to obtain
$$\left(1+\frac{a_1}{a_{i_1}}\right)\left(1+\frac{a_2}{a_{i_2}}\right)\cdot\ldots\cdot\left(1+\frac{a_n}{a_{i_n}}\right) \le (1+S)^n\ ,$$
where $S=\dfrac{1}{n}\left(\dfrac{a_1}{a_{i_1}}+\dfrac{a_2}{a_{i_2}}+\ldots+\dfrac{a_n}{a_{i_n}}\right)$. Now applying the rearrangement inequality we conclude that S attains its maximum, equal to 1, for the identity permutation, $i_1=1,\ i_2=2,\dots,i_n=n$. The proof is complete.

5.11. *If* a,b,c *are the sides of a triangle, then*
$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\le 3abc\ .$$
(Problem 2, IMO, 1964, proposed by Hungary)

Solution 1. By putting
$$b+c-a=x>0,\quad c+a-b=y>0,\quad a+b-c=z>0$$
we get
$$a=\frac{y+z}{2},\quad b=\frac{z+x}{2},\quad c=\frac{x+y}{2},$$
and the inequality to be proved becomes
$$2[(x+y)^2z+(y+z)^2x+(z+x)^2y]\le 3(x+y)(y+z)(z+x).$$
which can be written equivalently as
$$6xyz\le (y^2+z^2)x+(z^2+x^2)y+(x^2+y^2)z.$$
This last inequality is obvious by using the following AM-GM inequalities
$$y^2+z^2\ge 2yz,\quad z^2+x^2\ge 2zx,\quad x^2+y^2\ge 2xy.$$
The equality holds when $x=y=z,$ i.e., $a=b=c.$

Solution 2. Since a,b,c are the sides of a triangle we have

$$b+c-a>0,\ c+a-b>0,\ a+b-c>0$$
which together with the obvious inequalities
$$(a-b)^2 \geq 0,\ (b-c)^2 \geq 0,\ (c-a)^2 \geq 0$$
yields
$$(b-c)^2(b+c-a)+(c-a)^2(c+a-b)+(a-b)^2(a+b-c) \geq 0$$
which is equivalent to
$$6abc-2a^2(b+c-a)-2b^2(c+a-b)-2c^2(a+b-c) \geq 0$$
that is
$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \leq 3abc\,.$$

LINEAR ALGEBRA

Mircea Becheanu
ROMANIAN MATHEMATICAL SOCIETY

The methods of Linear Algebra are also important in elementary mathematics. Some elementary problems often have nice and unexpectedly easy solutions when they are modeled upon linear behavior. In such cases, the experience of the solver plays a leading role. We try to illustrate this by using a collection of selected problems in this area. We assume that the reader has a basic knowledge in Linear Algebra like: matrices, determinants, systems of linear equations, rank of a matrix, vector spaces and linear combinations of vectors.

We will present these basic problems which, in our opinion, are appropriate examples to illustrate our ideas.

1. A well known problem. Let m, n be positive integers which are relatively prime and $m < n$. There are n players around a game table. At some moment one can see that each group of m consecutive players have together the same amount, say s.Find the exact amount of each player.

Solution.

Let $x_1,x_2,...,x_n$ denote the amount of the player 1, 2,...,n, respectively. The condition given in the statement of the problem turns into the following equalities:

$$x_1 + x_2 + \cdots + x_m = s$$

$$x_2 + x_3 + \cdots + x_{m+1} = s$$

$$\dots\dots\dots$$

$$x_{n-1} + x_n + \cdots + x_{m-2} = s$$

$$x_n + x_1 + \cdots + x_{m-1} = s$$

This is a system of n linear equations which is satisfied by the numbers $x_1,x_2,...,x_n$. Because it is a square system, the general principles of Linear Algebra invite us to use Cramer's rule. So, we check informations about the determinant of the system, which is,

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 1 & \dots & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 0 & \ldots & 0 & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & 0 & 0 & \ldots & 1 \end{vmatrix}$$

It is not obvious how to compute this determinant or even to see whether $\Delta \neq 0$. Therefore it is not recommended to answer the problem by straightly solving the system.

A better attempt is to look carefully at the form of the system equations. It is easy to see that

$$x_1 = x_2 = \cdots = x_n = \frac{s}{m}$$

is a solution of the system. We will show that this solution is unique. To simplify the task, it is sufficient to show that the corresponding homogeneous system

$$\sum_{i=k}^{k+m-1} x_i = 0 \qquad \forall k = 1, \ldots, n$$

has only the trivial solution: $x_1 = x_2 = x_3 = \cdots = x_n = 0$ (here the indexes of the variables are taken modulo *n*, that $x_i + x_{n+1}, \ldots$etc.). To present clearly our method we start with a particular case, say *m* = 3 and *n* = 10. Then we have the equations:

$$\begin{aligned} &x_1 + x_2 + x_3 = 0 \\ &x_2 + x_3 + x_4 = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &x_{10} + x_1 + x_2 = 0 \end{aligned}$$

By adding the 10 equations and dividing the sum by 3 we obtain the sum:

$$x_1 + x_2 + \cdots + x_{10} = 0$$

In this sum we group together three consecutive variables and obtain:

$$(x_1 + x_2 + x_3) + \cdots + (x_7 + x_8 + x_9) + x_{10} = 0$$

It follows that $x_{10} = 0$.

The long summation can be permuted in a cyclic way, which yields

$$x_2 + x_3 + \cdots + x_{10} + x_1 = 0$$

and using the same method we obtain $x_1 = 0$. It is now clear how we can obtain that all variables are zero.

For the general case, by adding the equations of the system, like in the example, we obtain the long summation

$$x_1 + x_2 + \cdots + x_n = 0$$

and $x_k + x_{k+1} + \cdots + x_{k+m-1} = 0$ for all k =1, 2,...,n.

Let us consider Euclid's algorithm for the numbers n and m:

$$\begin{aligned}
&n = mq_1 + r_1;\ \ 0 < r_1 < m \\
&m = r_1q_2 + r_2;\ 0 < r_2 < r_1 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&r_{p-2} = r_{p-1}q_p + r_p;\ 0 < r_p < r_{p-1} \\
&r_{p-1} = r_pq_{p+1}
\end{aligned}$$

Since we have assumed m and n to be relatively prime, we know that
$r_p = g.c.d(m, n)$=1.

In the long summation we group successively m consecutive variables whose sum is zero. One obtains q_1 groups and there are remaining the last r_1 variables whose sum is zero. By cyclic permutations in the long summation, like in the example above, one obtains that the sum of any r_1 consecutive variables is zero (the index of variables being taken modulo n).

The procedure can be repeated by taking in any sum of m consecutive variables q_2 groups of r_1 variables and one obtains that the sum of r_2 consecutive variables is zero. Finally, because r_p =1, one gets that each variable is zero.

2. An IMO Problem with a past in older competitions. The next problem was proposed as a contest problem in 19th I.M.O, 1979. For many years after it was considered as a significant example about "what an IMO problem is"! The statement of the problem is:

Problem. Find the maximum length (number of terms) of a sequence $x_1, x_2, \ldots, x_n$ of real numbers such that the sum of any seven consecutive terms is negative and the sum of any eleven consecutive terms is positive.

Before we solve the problem we invite the reader to compare it with the following Russian olympiad problems (1969):

Problem 1.(8^{th} *Grade*). Is it possible to write in a row 20 numbers such that the sum of any three consecutive numbers is positive and the sum of all numbers is negative?

Problem 2.(9^{th} *Grade*). Is it possible to write in a row 50 real numbers such that the sum of any 17 consecutive numbers is positive and the sum of any 10 consecutive numbers is negative? The following solution will show that in all three cases we have essentially the same problem.

Solution. It is easy to see that $n < 77$. Indeed, in the sequence

$x_1, x_2, \ldots, x_7, \ldots, x_{11}, \ldots, x_{77}$

one may form 11 groups of seven consecutive terms and 7 groups of eleven consecutive terms. It follows that the sum of all terms is both positive and negative, which is a contradiction.

A finer argument shows that $n < 17$. Indeed, starting with a sequence $x_1, x_2, \ldots, x_{17}$, one may construct the following matrix:

$$\begin{array}{cccc} x_1 & x_2 & \ldots & x_7 \\ x_2 & x_3 & \ldots & x_8 \\ \vdots & \vdots & \ddots & \vdots \\ x_{11} & x_{12} & \ldots & x_{17} \end{array}$$

Taking the sum of the entries in the rows one obtains that the sum of the elements of the matrix is negative. Taking the sum of the entries in the columns one find that the sum of the elements of the matrix is negative. Again we get a contradiction!

We will show that there exist 16 numbers such that the sequence

$$x_1,\ x_2, \ldots, x_{16}$$

satisfies the conditions. During the IMO, the official solution just stated that the following sequence works:

5, 5, -13, 5, 5, 5, -13, 5, 5, -13, 5, 5, 5, -13, 5, 5.

Without giving a mathematical argument about how it appears we are far from considering it is a complete solution. In the following we will give two complete solutions. Until some point they are identical.

The conditions of the problem can be written in to the following 16×16 square system of linear equations:

$$
\begin{gathered}
x_1 + x_2 + \cdots + x_7 = a_1 \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{10} + x_{11} + \cdots + x_{16} = a_{10} \\
x_1 + x_2 + x_3 \cdots + x_{11} = b_1 \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
x_6 + x_7 + x_8 \cdots + x_{16} = b_6
\end{gathered}
$$

where $a_1, a_2, ..., a_{10} > 0$ and $b_1, b_2, ..., b_6 < 0$ are suitable real numbers. Therefore, the problem is transformed into the following: "are there positive numbers $a_1, ..., a_{10}$ and negative numbers $b_1, ..., b_6$ such that the system has a solution?"

Solution 1. The most favorable situation is when the determinant of the system is not zero. The experts can immediately recognize that in our case the determinant is the classical resultant $R(\Phi_7, \Phi_{11})$ of the polynomials

$$
\begin{gathered}
\phi_7 = X^{16} + X^5 + \cdots + X + 1 \\
\phi_{11} = X^{10} + X^9 + \cdots + X + 1
\end{gathered}
$$

which are the cyclotomic polynomials of degree 7 and 11, respectively. Since these polynomials have no common roots it follows that $R(\Phi_7, \Phi_{11}) \neq 0.$, so the system has a unique solution for any choice of the numbers $a_1, a_2, ..., a_{10}$ and $b_1, b_2, ..., b_6$.

The above solution uses some advanced theoretical background, like the theory of resultant of two polynomials. It is interesting to remark that the problem has also an elementary solution.

Solution 2. Denote $y_k = x_1 + x_2 + \ldots + x_k$ for all $k = 1, 2, ..., 16$. The new variables

$y_1, y_2, ..., y_{16}$ are connected to former variables $x_1, x_2, ..., x_{16}$ through a non singular linear transformation given by the following triangular matrix:

$$
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & 1 & 0 & \ldots & 0 \\
1 & 1 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_1 \\ x_1 \\ \vdots \\ x_1
\end{pmatrix}
=
\begin{pmatrix}
y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_4
\end{pmatrix}
$$

Therefore, it is suffcient to find $y_1, y_2, ..., y_{16}$ such that the following inequalities hold:

$$y^{7} < 0;\ y_8 - y_1 < 0;\ y_9 - y_2 < 0;\ y_{10} - y_3 < 0;\ y_{11} - y_4 < 0;\ \ldots;\ y_{16} - y_9 < 0,$$
$$y_{11} > 0;\ y_{12} - y_1 > 0;\ y_{13} - y_2 > 0;\ \ldots;\ y_{16} - y_5 > 0 \quad .$$

Here there are 16 inequalities in all. They can be arranged in the following increasing sequence of inequalities:

$$y_{10} < y_3 < y_{14} < y_7 < 0 < y_{11} < y_4 < y_{15} < y_8 < y_1 < y_{12} < y_5 < y_{16} < y_9 < y_2 < y_{13} < y_6.$$

The conclusion is: by taking the numbers $y_1, y_2, \ldots, y_{16}$ in that order, one may find the numbers $x_1, x_2, \ldots, x_{16}$ which satisfy the conditions required by the problem. And this task can be easily fulfiled.

3. Linearity has unexpected applications. The next problem comes from the 40^{th} Russian Olympiad,1988. The statement of the problem has no connection with linearity at all. For any real number *x* and any sequence

$a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ denote

$$T_x(a) = \left(|a_1 - x|, |a_2 - x|, \ldots, |a_n - x|\right)$$

In this way, one obtains a transformation

$$T_x : \mathbb{R}^n \to \mathbb{R}^n,$$
$$a \to T_x(a)$$

(*i*) Show that for any $a \in \mathbb{R}^n$ there exists a finite sequence of transformations of this type which, by applying them successively, map a into the zero-sequence
0 = (0, 0,..., 0).

(*ii*) Find the smallest number of transformations which are necessary to transform an arbitrary sequence into 0 = (0, 0,..., 0).

Solution. In the first part we will present the strategy how can we map a sequence into the zero sequence. First, let us remark that if the composition $T_k \circ \cdots \circ T_1$ maps *a* in 0 then $T_{k-1} \circ \cdots \circ T_{1(a)}$ has all its components equal. Another useful remark is the following: if a sequence *a* has some components equal, then $T_x(a)$ has the same property.

As a consequence of these remarks we create the following strategy: find transformations which produce equal components. To begin, choose $x_1 = \dfrac{a_1 + a_2}{2}$ Then T_{x_1} transforms $a = (a_1, a_2, \ldots, a_n)$ into $b = (b_1, b_2, \ldots, b_n)$, where

$$b_1 = |a_1 - x_1| = \left|a_1 - \frac{a_1 + a_2}{2}\right| = \left|\frac{a_1 - a_2}{2}\right| = \left|a_2 - \frac{a_1 + a_2}{2}\right| = b_2$$

In the same way, choose $x_2 = \frac{b_2 + b_3}{2}$. Then, one gets $T_{x_2} \circ T_{x_1}(a) = c$, where $c_1 = c_2 = c_3$. Therefore, after such $n - 1$ transformations one obtains a sequence whose terms are all equal and after n transformations the result is a sequence with all components zero.

For the second part of the problem we will prove that the smallest number of transformations necessary to map an arbitrary sequence into 0 is n. To do this it is sufficient to show that there exists a sequence a which can not be mapped into 0 by only $n - 1$ transformations. This sequence is $f_n = (1!, 2!, ..., n!)$. The proof of the last statement will be given by induction on
n. We need the following lemma:

Lemma 1. Assume that the composition of transformations $T_{x1}, \cdots, T_{xk}$ maps the sequence $a = (a_1, a_2, ..., a_n)$ into 0 and that it contains a transformation T_{x_i} such that x_i is not greater than any component $c_1, c_2, ..., c_n$ of the sequence $T_{xi-1} \circ \cdots \circ, \cdots, T_{x1}(a)$ = $(c_1, c_2, ..., c_n)$. Then the number k of transformations can be minimized.

Proof of Lemma 1. Since $c_j \geq x_i$ for all $j = 1, 2, ..., n$ we have the equalities:

$$\left\|c_j - x_i\right| - x_{i+1}\right| = \left|c_j - \left(x_i + x_{i+1}\right)\right|,$$

for all $j \geq 1$. They can be translated into the following equality of composition of transformations:

$$T_{x-k} \circ \cdots \circ T_{x_i+1} \circ T_{x_i} \circ \cdots \circ T_{x_i} = T_{x_k} \circ \cdots \circ T_{x_i+1+x_i} \circ \cdots \circ T_{x_i}$$

This ends the proof of Lemma.
Going back to the problem, it is clear that the sequence f_1 can be mapped into 0 by one transformation. Assume that the sequence f_n can be mapped into zero by not less than n transformations and suppose by contradiction that the sequence f_{n+1} is mapped into 0 by n transformations. Then, the same transformations map the first its n components into 0. These components are $1!, 2!, ..., n!$, that is the sequence f_n.

By the induction assumption about the *sequence* f_n it follows that $1 \leq x_1 \leq n!$, and in each of the next steps the numbers $x_2, ..., x_n$ are not greater than the greatest of the component which appear. Hence, we have

$$|\ldots||(n+1)!-x_1|-x_2|\ldots-x_n|=(n+1)!-(x_1+x_2+\cdots+x+n)\ge(n+1)!-n.n!>0,$$

which is a contradiction.

Remarking the ingenuity of the last argument, we will give an alternative but more natural proof for the second part of the problem. In a very unexpected way, Linear Algebra will be helpful.

Assume by contradiction that for every sequence *a* = *(a_1,a_2,...,a_n)* there exist *n -1* transformations $T_{x_{n-1}},\ldots,T_{x-1}$ such that

$$T_{x_{n-1}}\circ\cdots\circ T_{x1}(a)=0$$

for all sequences *a*. This is equivalent with the following statement: for every real numbers a_1,a_2,...,a_n there exist *n - 1* numbers x_1,...,x_{n-1} and there exist suitable

n x (*n* - 1) numbers $\varepsilon_{ij}\in\{+1,-1\},1\le i\le n,1\le j\le n-1$, such that the following equalities hold:

$$\begin{aligned}
&\varepsilon_{11}x_1+\varepsilon_{12}x_2+\ldots+\varepsilon_{1n-1}x_{n-1}=a_1\\
&\varepsilon_{21}x_1+\varepsilon_{22}x_2+\ldots+\varepsilon_{2n-1}x_{n-1}=a_2\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\varepsilon_{n1}x_1+\varepsilon_{n2}x_2+\ldots+\varepsilon_{nn-1}x_{n-1}=a_n
\end{aligned}$$

This is a system of *n* linear equations in *n - 1* variables whose entries are in the set $\{+1,-1\}$ In the language of geometry, x_1,...,x_{n-1} is a solution of the system if and only if the vector *a* = *(a_1,a_2,...,a_n)* lies in the subspace of $\mathbb{R}^n$ spanned by the

n - 1 columns of the matrix of the system. The number of the systems of this kind is finite. Therefore, the result obtained above can be restated as follows: every vector $a\in\mathbb{R}^n$ lies in a finite union of subspaces of dimension at most *n* - 1. This is impossible, as it follows from the following lemma:

Lemma 2. The linear space $\mathbb{R}^n$ can not be represented as a finite union $\mathbb{R}^n=V_1\cup\cdots\cup V_m$ of subspaces of dimension *n – 1*

Proof of the Lemma 2. Every subspace V_i of dimension *n – 1 is* the set of solutions of a non-zero linear form $F_i(x)=\sum_{k=1}^n a_kx_k=0$. The polynomials F_k are not zero, so that the polynomial $F(x)=\prod_{k=1}^m F_k(x)$ is no zero. It is easy to see by induction on the number n of variables that a non zero real polynomial can not vanish on the whole space $\mathbb{R}^n$. This ends the proof.

NUMBER THEORY

Jenő Szigeti
University of Miskolc

1. DIVISIBILITY

We state four fundamental results concerning divisibility, without proofs. Let $\varphi(n)$ denote the number of elements in $\{1, 2, ..., n\}$ relative prime to n. For example if $p \geq 2$ is a prime number, then $\varphi(p) = p - 1$.

1.1. Theorem (Euler). *Let $a \in \mathbb{Z}$ be an integer relative prime to n. Then $a^{\varphi(n)} - 1$ is divisible by n:*

$$n \mid a^{\varphi(n)} - 1$$

1.2. Theorem (Fermat). *Let $a \in \mathbb{Z}$ be an integer not divisible by a given prime number p. Then $a^{p-1}-1$ is divisible by p:*

$$p \mid a^{p-1} - 1.$$

1.3. Theorem (Wilson). *If $p \geq 2$ is a prime number, then $(p-1)!+1$ is divisible by p:*

$$p \mid (p-1)! + 1$$

1.4. Theorem (Chinese Remainder). *If $a_1, a_2, ..., a_n \in \mathbb{Z}$ and $r_1, r_2, ..., r_n \in \mathbb{Z}$ are integers such that $\gcd(a_i, a_j) = 1$ for all $1 \leq i < j \leq n$, then there exists an integer $b \in \mathbb{Z}$ with the property that $a_i \mid b - r_i$ for all $1 \leq i \leq n$.*

1.5. Example. *Prove that $5^n \mid 2^k - 1$ if $k = 5^n - 5^{n-1}$ and $5^n \nmid 2^k - 1$ if $1 \leq k < 5^n - 5^{n-1}$*

Solution. Since $\varphi(5^n) = 5^n - 5^{n-1} = 4 \cdot 5^{n-1}$ the application of Euler's theorem gives that $5^n \mid 2^{5^n - 5^{n-1}} - 1$. We proceed by induction. For $n = 1$ and $k = 5^1\text{-}5^0 = 4$ we have $5 \mid 2^4-1 = 15$, moreover for $1 \leq k < 5^1\text{-}5^0 = 4$ it is easy to see that $5 \nmid 2^k-1$. Assume that our statement is valid for n and fails to hold for $n + 1$.
Let $1 \leq k < 5^{n+1}\text{-}5^n$ be the smallest k with $5^{n+1} \mid 2^k-1$. Now $4 \cdot 5^n = 5^{n+1}-5^n = kq + r$, where $0 \leq r \leq k - 1$ is the reminder of the division by k. If $r \geq 1$ then

$$5^{n+1} \mid 2^{5^{n+1}-5^n} - 1 = 2^{kq+r} = ((2^k)^q - 1)2^r + (2^r - 1)$$

and $2^k-1 \mid (2^k)^q-1$ would imply that $5^{n+1} \mid 2^r-1$ in contradiction with the choice of k. Thus we have $r = 0$, i.e. that $4 \cdot 5^n = kq$. We claim that $4 \cdot 5^{n-1}$ is a divisor of k. Let $k = 4 \cdot 5^{n-1}t + m$, where $0 \leq m < 4 \cdot 5^{n-1} = 5^n - 5^{n-1}$ is the reminder of the division by $4 \cdot 5^{n-1}$. If $m \geq 1$ then

$$5^{n+1} \mid 2^k - 1 = 2^{4 \cdot 5^{n-1}t+m} - 1 = ((2^{4 \cdot 5^{n-1}})^t - 1)2^m + (2^m - 1)$$

and $2^{4\cdot5^{n-1}}-1\,|\,(2^{4\cdot5^{n-1}})^t-1$ would imply that $5^n \mid 2^m - 1$ in contradiction with our hypothesis on n. Thus we have $m = 0$, i.e. that $k = 4 \cdot 5^{n-1} t$. In view of $4 \cdot 5^n = kq = 4 \cdot 5^{n-1} tq$, we deduce that $t = 1$ or $t = 5$. The case $t = 1$ is impossible because of $5^{n+1} \nmid 2^{4\cdot5^{n-1}}-1=2^k-1$.

The case $t = 5$ is also impossible because of $k = 4 \cdot 5^n \not< 5^{n+1}-5^n$. We only have to show that $5^{n+1} \nmid 2^{4\cdot5^{n-1}}-1$. Now $5^{n-1}\,|\,2^{5^{n-1}-5^{n-2}}-1=2^{4\cdot5^{n-2}}-1$ and $5^n \nmid 2^{4\cdot5^{n-2}}-1$ follows from our hypothesis on n. In consequence, we get that $2^{4\cdot5^{n-2}}-1=5^{n-1}s$, where $5 \nmid s$. Thus

$$2^{4\cdot5^{n-1}}-1=(5^{n-1}s+1)^5-1=$$
$$=(5^{n-1}s)^5+5(5^{n-1}s)^4+10(5^{n-1}s)^3+10(5^{n-1}s)^2+5(5^{n-1}s)=5^{n+1}w+5^n s$$

is not divisible by 5^{n+1}.

1.6. Example. *Prove that* $p\,\left|\,\left[\left(\dfrac{p-1}{2}\right)!\right]^2+1\right.$, *where* $p = 4k+1$ *is a prime number.*

Solution. We have $p \mid (p-1)!+1=1 \cdot 2 \cdot \ldots \cdot (2k) \cdot (2k+1) \cdot \ldots \cdot (p-1) + 1$ by Wilson's theorem. Since

$$1\cdot2\cdot\ldots\cdot(2k)\cdot(2k+1)\cdot\ldots\cdot(p-1)+1 = 1\cdot2\cdot\ldots\cdot(2k)\cdot(p-2k)\cdot(p-(2k-1))\cdot\ldots\cdot(p-1)+1=$$
$$= [(2k)!]^2 + pw + 1,$$

for some integer w, we obtain the desired divisibility.

1.7. Example. *Prove that* $p = x^2 + y^2$ *has an integer solution* (x, y), *where* $p = 4k + 1$ *is a prime number.*

Solution. If $mp = x^2 + y^2$ is solvable for some $2 \le m < p$ (this is our case by Example 1.6), then we prove that $np = x^2 + y^2$ is also solvable for some $1 \le n \le \dfrac{m}{2}$. If m is even, then

$$\frac{m}{2}p=\left(\frac{x+y}{2}\right)^2+\left(\frac{x-y}{2}\right)^2,$$

where $\dfrac{x+y}{2}$ and $\dfrac{x-y}{2}$ are integers. If m is odd, then we have

$$x = mr + x_1 \text{ and } y = ms + y_1$$

for some integers $|x_1| < \dfrac{m}{2}$ and $|y_1| < \dfrac{m}{2}$. Now

$$mp=x^2+y^2=(mr+x_1)^2+(ms+y_1)^2=mw+x_1^2+y_1^2$$

implies that $x_1^2+y_1^2=mn$ for some $n \ge 1$ ($n = 0$ is impossible, because of $mp = x^2 + y^2$ is not divisible by m^2). Using $|x_1| < \dfrac{m}{2}$ and $|y_1| < \dfrac{m}{2}$, we obtain that

$$mn=x_1^2+y_1^2<\left(\frac{m}{2}\right)^2+\left(\frac{m}{2}\right)^2=\frac{m^2}{2}$$

whence $n < \frac{m}{2}$ can be derived. In order to see that $np = x^2 + y^2$ is solvable consider the following

$$m^2np = (mp)(mn) = (x^2+y^2)(x_1^2+y_1^2) = (xx_1+yy_1)^2 + (xy_1 - x_1y)^2,$$

where

$$xx_1 + yy_1 = (mr + x_1)x_1 + (ms + y_1)y_1 = mrx_1 + msy_1 + x_1^2 + y_1^2 = m(rx_1 + sy_1 + n)$$

and

$$xy_1 - x_1y = (mr + x_1)y_1 - x_1(ms + y_1) = m(ry_1 - x_1s).$$

Thus we have

$$np = (rx_1 + sy_1 + n)^2 + (ry_1 - x_1s)^2.$$

An iterated application of the above descending argument (replacing m by $n \le \frac{m}{2}$) finally gives that $p = x^2 + y^2$ is solvable.

1.8. Problem. *Prove the following*

(1) $100 \mid 11^{10} - 1$
(2) $13 \mid 2^{70} + 3^{70}$
(3) $11 \cdot 31 \cdot 61 \mid 20^{15} - 1$
(4) $7 \mid 2222^{5555} + 5555^{2222}$
(5) $35 \mid 3^{6n} - 2^{6n}$
(6) $56486730 \mid mn(m^{60} - n^{60})$
(7) $169 \mid 3^{3n+3} - 26n - 27$
(8) $19 \mid 2^{2^{6k+2}} + 3$

Hints.

(1) $11^{10} - 1 = (11 - 1)(11^9 + 11^8 + \ldots + 11 + 1)$
(2) $13 \mid 2^{12} - 1$ implies $13 \mid 2^{60} - 1$ and $13 \mid 2^5 - 6$ implies $13 \mid 2^{10} + 3$. Thus we have $13 \mid 2^{70} + 3$. On the other hand $13 \mid 3^3 - 1$ implies $13 \mid 3^{69} - 1$ and $13 \mid 3^{70} - 3$.
(3) $11 \mid 2^5 + 1$ and $11 \mid 10^5 + 1$ imply $11 \mid 20^5 - 1$ and $11 \mid 20^{15} - 1$. $31 \mid 20^3 - 2$ implies $31 \mid 20^{15} - 2^5$, whence $31 \mid 20^{15} - 1$ follows. $61 \mid 3^4 - 20$ implies $61 \mid 3^{60} - 20$ and using $61 \mid 3^{60} - 1$ we get $61 \mid 20^{15} - 1$.
(4) $2222^{5555} + 5555^{2222} = (2222^{5555} + 4^{5555}) + (5555^{2222} - 4^{2222}) - (4^{5555} - 4^{2222})$, $4^{5555} - 4^{2222} = 4^{2222}(4^{3333} - 1)$, $4^{3333} - 1 = 64^{1111} - 1$ and $7 \mid 2222 + 4$, $7 \mid 5555 - 4$,
$7 \mid 64 - 1$.
(5) $3^3 + 2^3 \mid 3^6 - 2^6$ and $3^6 - 2^6 \mid 3^{6n} - 2^{6n}$.
(6) $56786730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 61$ and use Fermat's theorem.
(7) Apply induction: $(3^{3(n+1)+3} - 26(n+1) - 27) - (3^{3n+3} - 26n - 27) = 26(3^{3n+3} - 1)$ and
$13 \mid 3^3 - 1$.
(8) $18 \mid 2^{6k+2} - 4$ implies $2^{2^{6k+2}} + 3 = 2^{18t+4} + 3 = 2^4(2^{18t} - 1) + 19$ and $19 \mid 2^{18} - 1$.

1.9. Problem. *The last four digits of a square number are equal. Find this digit.*

Hint. The last digit of a square can only be 0, 1, 4, 5, 6 and 9. However the last two digits of a square can not be 11, 55, 66 and 99 (divide by 4). Of the remaining cases note that the last four digits can not be 4444 (divide by 16). So the only possibility is 0000 which comes, for example, from 100^2.

1.10. Problem. *Prove that* $(n!)^{(n-1)!}$ *is a divisor of* $(n!)!$.

Hint. Use that $n! \mid (t + 1)(t + 2)\ldots(t + n)$.

1.11. Problem. *Let* $m \geq 1$ *be an integer number. Prove that any even number can be represented as a difference of two integers being relative prime to m.*

Hint. Let $2k$ be the given even number and let $p_1, p_2, \ldots, p_r$ be the prime factors of m. For each index $1 \leq i \leq r$ there exists an integer x_i such that $f(x_i) = x_i(x_i + 2k)$ is not divisible by p_i. The Chinese Reminder theorem ensures the existence of an integer x with $p_i \mid x - x_i$ for all i. It follows that $p_i \mid f(x) - f(x_i)$ and $p_i \nmid f(x)$ for all i (here $f(x) = x(x + 2k)$). Thus $2k = (x + 2k) - x$ is a representation.

2. THE LINEAR EQUATION $ax + by = c$

2.1. Theorem. *Let a,b,c be non zero integer numbers. Then the following are equivalent:*

(1) *There exist integer numbers x and y such that* $ax + by = c$ *holds.*

(2) *The greatest common divisor of a and b is a divisor of c, namely* $\gcd(a, b) \mid c$.

If (x_0,y_0) *is an arbitrary solution of* $ax + by = c$, *then any other solution* (x, y) *can be obtained as*

$$x = x_0 + b_0t \text{ and } y = y_0 - a_0t,$$

where t is an integer and $a_0 = a/\gcd(a,b)$ *and* $b_0 = b/\gcd(a, b)$.

Proof. Let $d = \gcd(a, b)$.

<u>(1)⇒(2)</u> Now $d \mid a$ and $d \mid b$ imply $d \mid ax$ and $d \mid by$, whence $d \mid ax + by$ can be derived. Thus we have $d \mid c$.

<u>(2)⇒(1)</u> Now $c_0 = c/d$ is an integer number and $ax + by = c$ is equivalent to $a_0x + b_0y = c_0$, where $\gcd(a_0, b_0) = 1$. As we have already seen in a lemma (using the Euclidean algorithm) $\gcd(a_0,b_0) = 1$ implies that $a_0x' + b_0y' = 1$ holds for some pair (x',y') of integers. Clearly, (x',y') will be a solution of $ax + by = d$ and (c_0x',c_0y') will be a solution of $ax + by = c$.

In order to produce all of the solutions (x, y) of $ax + by = c$ from $ax_0 + by_0 = c$ consider the following

$$ax + by = ax_0 + by_0 .$$

First $a(x - x_0) = b(y_0 - y)$ and next $a_0(x - x_0) = b_0(y_0 - y)$ can be obtained from the above equation. Since $\gcd(a_0, b_0) = 1$, we deduce that $b_0 \mid x - x_0$ i.e. that $x - x_0 = b_0t$ holds for some integer t. Now $a_0b_0t = b_0(y_0 - y)$ implies that $y_0 - y = a_0t$. Thus we have $x = x_0 + b_0t$ and
$y = y_0 - a_0t$. It is easy to check, that for any choice of the integer t, the pair $(x_0 + b_0t, y_0 - a_0t)$ is a solution of $ax + by = c$.

2.2. Example. *Find the (integer) solutions of the equation* $354x + 138y = 12$.

Solution. Now gcd(354, 138) = 6 and $354 = 6 \cdot 59$, $138 = 6 \cdot 23$, $12 = 6 \cdot 2$. As $6 \mid 12$, the equation is solvable in integer numbers. It is enough to deal with $59x + 23y = 2$, where gcd(59, 23) = 1. First we solve $59x + 23y = 1$ by using the Euclidean algorithm for the pair (59, 23). The steps are the following:

$$59 = 2 \cdot 23 + 13\ ,\ 23 = 1 \cdot 13 + 10\ ,\ 13 = 1 \cdot 10 + 3\ ,\ 10 = 3 \cdot 3 + 1\ ,\ 3 = 3 \cdot 1 + 0.$$

We obtain the following expressions for the consecutive remainders:

$$\begin{aligned}
13 &= 59 - 23 \cdot 2 = 59 \cdot 1 + 23 \cdot (-2),\\
10 &= 23 - 13 \cdot 1 = 23 - (59 \cdot 1 + 23 \cdot (-2)) = 59 \cdot (-1) + 23 \cdot 3,\\
3 &= 13 - 10 \cdot 1 = (59 \cdot 1 + 23 \cdot (-2)) - (59 \cdot (-1) + 23 \cdot 3) = 59 \cdot 2 + 23 \cdot (-5)\\
1 &= 10 - 3 \cdot 3 = (59 \cdot (-1) + 23 \cdot 3) - (59 \cdot 2 + 23 \cdot (-5)) \cdot 3 = 59 \cdot (-7) + 23 \cdot 18.
\end{aligned}$$

Thus $x' = -7$ and $y' = 18$ is a solution of $59x + 23y = 1$. Clearly, $x_0 = (-7)\cdot 2 = -14$ and $y_0 = 18 \cdot 2 = 36$ is a solution of $59x + 23y = 2$. Using Theorem 2.1, we obtain all of the solutions of $59x + 23y = 2$ as

$$(-14 + 23t, 36 - 59t),$$

where t is an arbitrary integer number.

2.3. Example. *Find the (integer) solutions of the equation* $35x + 15y + 21z = 8$.

Solution. An equivalent form of the equation is $5(7x + 3y) + 21z = 8$. First we solve $5u + 21z = 8$. It is easy to see that $u' = -4$ and $z' = 1$ is a solution of $5u + 21z = 1$, it follows that $(-32, 8)$ is a solution of $5u + 21z = 8$. The use of our theorem gives that the solutions of $5u + 21z = 8$ can be written as $(-32 + 21t, 8 - 5t)$, $t \in \mathbb{Z}$. Now $5(7x + 3y) + 21z = 8$ holds for the integers x, y, z if and only if $7x + 3y = -32 + 21t$ and $z = 8 - 5t$ for some integer t. The only remaining problem is to find the solutions of $7x + 3y = -32 + 21t$ for all t. The pair $(1, -2)$ is a solution of $7x + 3y = 1$, thus $(-32 + 21t, 64 - 42t)$ is a solution of $7x + 3y = -32 + 21t$. By Theorem 2.1, the general solution is of the form

$$x = -32 + 21t + 3s \text{ and } y = 64 - 42t - 7s,$$

where s is an arbitrary integer. In consequence (x, y, z) is a solution of $35x + 15y + 21z = 8$ if and only if we can find integers t and s such that

$$(x, y, z) = (-32 + 21t + 3s, 64 - 42t - 7s, 8 - 5t).$$

2.4. Problem. *Find the (integer) solutions of the following system of linear equations*

$3x + 2y - 7z = 5$, $2x - 5y + 9z = 2$.

3. NON LINEAR DIOPHANTINE EQUATIONS

3.1. Theorem. *Let x, y, z be positive (non zero) integer numbers and $x = dx_1$, $y = dy_1$, $z = dz_1$, with $d = \gcd(x, y, z) > 0$. Then we have $x^2 + y^2 = z^2$ if and only if (1) or (2) holds.*

(1) $x_1 = u^2 - v^2$, $y_1 = 2uv$ *and* $z_1 = u^2 + v^2$ *for some integers* $u > v \geq 1$ *with* $\gcd(u, v) = 1$.
(2) $x_1 = 2uv$, $y_1 = u^2 - v^2$ *and* $z_1 = u^2 + v^2$ *for some integers* $u > v \geq 1$ *with* $\gcd(u, v) = 1$.

Proof. $x^2 + y^2 = z^2$ can be written as $d^2x_1^2 + d^2y_1^2 = d^2z_1^2$, thus we can cancel by d^2. Now $\gcd(x_1, y_1) = 1$, $\gcd(x_1, z_1) = 1$, $\gcd(y_1, z_1) = 1$. We claim first that exactly one of x_1 and y_1 is even. Indeed, the condition $\gcd(x_1, y_1) = 1$ shows that at most one of x_1 and y_1 is even. On the other hand if both x_1 and y_1 where odd, $x_1 = 2k + 1$ and $y_1 = 2l + 1$, we would have

$$z_1^2 = x_1^2 + y_1^2 = 4(k^2 + k + l^2 + l) + 2 \qquad (*)$$

so that z_1^2 and hence z_1 is even. But then $4 \mid z_1^2$, which contradicts (*). This completes the argument that exactly one of x_1, y_1 is even.
Suppose that $y_1 = 2l$ is even, so that x_1, and hence z_1 as well, are odd numbers. Now

$$(z_1 + x_1)(z_1 - x_1) = z_1^2 - x_1^2 = y_1^2$$

implies that

$$\left(\frac{z_1 + x_1}{2}\right)\left(\frac{z_1 - x_1}{2}\right) = \left(\frac{y_1}{2}\right)^2.$$

Since

$$\frac{z_1 + x_1}{2} - \frac{z_1 - x_1}{2} = x_1, \quad \frac{z_1 + x_1}{2} + \frac{z_1 - x_1}{2} = z_1$$

and $\gcd(x_1, y_1) = 1$, we obtain that

$$\gcd\left(\frac{z_1 + x_1}{2}, \frac{z_1 - x_1}{2}\right) = 1$$

The product of two relatively prime integer numbers $\frac{z_1 + x_1}{2}$ and $\frac{z_1 - x_1}{2}$ is a square number, it follows that each factor is a square, i.e. that

$\frac{z_1 + x_1}{2} = u^2$ and $\frac{z_1 - x_1}{2} = v^2$

hold for some integers $u > v \geq 1$, where $\gcd(u, v) = 1$. In view of the above equations, first we deduce that

$$u^2 - v^2 = x_1 \text{ and } u^2 + v^2 = z_1$$

and then

$$y_1^2 = z_1^2 - x_1^2 = \left(u^2 + v^2\right)^2 - \left(u^2 - v^2\right)^2 = 4u^2v^2$$

gives that $y_1 = 2uv$. The case when x_1 is even can be treated similarly and will provide the situation described in (2).
Finally we note that

$$\left(u^2 - v^2\right)^2 + (2uv)^2 = \left(u^2 + v^2\right)^2$$

is an identity.

3.2. Example. *Find the (integer) solutions of the equation* $12x^2 + y^2 = z^2$.

Solution. We are interested in the non negative solutions. Cancel by d^2 with $d = \gcd(x, y, z)$ and let $12x_1^2 + y_1^2 = z_1^2$ be the resulting equation. Now y_1 and z_1 are of the same parity, moreover $\gcd(x_1, y_1, z_1) = 1$. Clearly,

$$3x_1^2 = \left(\frac{z_1 + y_1}{2}\right)\left(\frac{z_1 - y_1}{2}\right)$$

and take $\gcd\left(\frac{z_1 + y_1}{2}, \frac{z_1 - y_1}{2}\right) = \delta$. Then we have either

$$\frac{z_1 + y_1}{2} = 3u^2\delta \text{ and } \frac{z_1 - y_1}{2} = v^2\delta$$

or

$$\frac{z_1 + y_1}{2} = v^2\delta \text{ and } \frac{z_1 - y_1}{2} = 3u^2\delta$$

for some relatively prime integers $u \geq 0$ and $v \geq 0$. In the first and second cases we obtain that

$$z_1 = \left(3u^2 + v^2\right)\delta, \; y_1 = \left(3u^2 - v^2\right)\delta, \; x_1 = uv\delta$$

and

$$z_1 = \left(3u^2 + v^2\right)\delta, \; y_1 = \left(v^2 - 3u^2\right)\delta, \; x_1 = uv\delta$$

respectively. Since $\delta \mid x_1$, $\delta \mid y_1$ and $\delta \mid z_1$ in both cases, we get that $\delta = 1$. The non negative solutions are

$$x_1 = uv \ , \ y_1 = |3u^2 - v^2| \ , \ z_1 = 3u^2 + v^2 \ .$$

3.3. Theorem. *If* (x, y, z) *is an integer solution of* $x^4 + y^4 = z^2$ *then* $x = 0$ *or* $y = 0$.

Proof. Let $x \geq 1$, $y \geq 1$, $z \geq 1$ and (x, y, z) be a solution such that z is the smallest possible. Without loss of generality $\gcd(x, y) = 1$ can be assumed. We can apply Theorem 3.1 to get

$$x^2 = 2uv \ , \ y^2 = u^2 - v^2 \text{ and } z = u^2 + v^2$$

for some integers $u > d \geq 1$ with $\gcd(u, v) = 1$. In view of $x^2 = 2uv$, we get that one of u and v is even (the other one is odd). If $u = 2k$ and $v = 2l + 1$ then $y^2 = u^2 - v^2 = 4(k^2 - l^2 - l) - 1$, a contradiction. Hence $v = 2l$ and $x^2 = 4ul$, whence we obtain

$$\left(\frac{x}{2}\right)^2 = ul$$

Since $\gcd(u, l) = 1$, we can write that $u = z_1^2$ and $l = v_1^2$ for some integers z_1 and v_1. Now $v = 2v_1^2$ and z_1 is odd. We can rewrite $y^2 = u^2 - v^2$ as follows

$$\left(2v_1^2\right)^2 + y^2 = \left(z_1^2\right)^2,$$

where $\gcd(2v_1^2, y) = 1$. The repeated application of Theorem 3.1 gives that

$$2v_1^2 = 2u_2v_2 \ , \ \ y = u_2^2 - v_2^2 \ , \ z_1^2 = u_2^2 + v_2^2$$

for some integers $u_2 > v_2 \geq 1$ with $\gcd(u_2, v_2) = 1$. Now $v_1^2 = u_2v_2$ implies that $u_2 = x_1^2$ and $v_2 = y_1^2$ for some relatively prime integers x_1 and y_1. We can rewrite $z_1^2 = u_2^2 + v_2^2$ as follows

$$x_1^4 + y_1^4 = z_1^2$$

Now (x_1, y_1, z_1) is a solution, where $z_1 \leq z_1^2 = u < u^2 + v^2 = z$ in contradiction with the choice of z.

3.4. Example. *Find the (integer) solutions of the equation* $2x^6 + 3y^6 = z^6$.

Solution. Let $(x,y) \neq (0,0)$. Without loss of generality we can assume that one of x and y is not divisible by 7 (otherwise we can cancel by 7^6). We distinguish three cases according to whether 7 divides one or none of x and y .

- If $7 \nmid x$ and $7 \nmid y$ then $7 \mid 2(x^6 - 1) + 3(y^6 - 1) = z^6 - 5$ by Fermat's theorem. Since $7 \mid z$ or

$7 \mid z^6 - 1$, we obtain that $z^6 - 5$ can not be divisible by 7, a contradiction.
- If $7 \mid x$ and $7 \nmid y$ then $7 \mid 2x^6 + 3(y^6 - 1) = z^6 - 3$. Since $7 \mid z$ or $7 \mid z^6 - 1$, we obtain that $z^6 - 3$ can not be divisible by 7, a contradiction.
- If $7 \nmid x$ and $7 \mid y$ then $7 \mid 2(x^6 - 1) + 3y^6 = z^6 - 2$. Since $7 \mid z$ or $7 \mid z^6 - 1$, we obtain that $z^6 - 2$ can not be divisible by 7, another contradiction.

Thus the only solution is $x = y = z = 0$.

One of the most important results in the general theory of Diophantine equations is the following.

3.5. Theorem (Roth, 1955). *Let* $a_0, a_1, \ldots, a_n, b_{ij} \in \mathbb{Z}$ $(0 \le i, j)$ *be integer numbers such that the polynomial* $f(x) = a_0 + a_1x + \ldots + a_nx^n$ *is irreducible over* $\mathbb{Z}$. *If* $n \ge 3$, *then the following equation has only a finite number of integer solutions*

$$a_nx^n + a_{n-1}x^{n-1}y + \ldots + a_1xy^{n-1} + a_0y^n = \sum_{i+j \le n-3} b_{ij}x^iy^j$$

3.6. Example. *The number of pairs* (x, y) *such that* $x, y \in \mathbb{Z}$ *and*

$$x^5 + 3x^3y^2 - 3x^2y^3 + 6y^5 = x^2 - 2y^2 + xy + 7x + 5y + 1$$

is finite.

Solution. Since $f(x) = x^5 + 3x^3 - 3x^2 + 6$ is irreducible over $\mathbb{Z}$ (because the only candidate integer roots $\pm1, \pm2, \pm3, \pm6$ fail to be roots), we can apply the above Theorem 3.5.

3.7. Problem. *Find the (integer) solutions of the equation*

$$2x^{12} + 3y^{12} + 4z^{12} = 11u^{12}$$

Hint. Use Fermat's theorem: $13 \mid n$ or $13 \mid n^{12} - 1$ for all integers n.

3.8. Problem. *Find all pairs* (x, y) *of integers such that*

$$x^2 + 3xy + 4006(x + y) + 2003^2 = 0.$$

Hint. Rewrite the equation as

$$-9y = 3x + 8012 + \frac{2003^2}{3x + 4006}$$

and use that 2003 is prime.

3.9. Problem. *Find all triples* (x, y, z) *of integers such that*

$$x + y + z = 3 \text{ and } x^3 + y^3 + z^3 = 3.$$

Hint. Use that $(x + y + z)^3 - (x^3 + y^3 + z^3) = 3(x + y)(x + z)(y + z)$.

3.10. Problem. *Let $n \geq 1$ be an integer and $p \geq 2$ be a prime number. Prove that $x(x + 1) = p^{2n}y(y + 1)$ has no integer solution with $x \geq 1$ and $y \geq 1$.*

Hint. Use $x + 1 \geq p^{2n}$ and $p^{2n} - 1 = [p^n(2y + 1) + (2x + 1)][p^n(2y + 1) - (2x + 1)]$.

3.11. Problem. *Let $D = m^2 + 1$ for some integer $m \geq 1$. Prove that $x^2 - Dy^2 = 1$ has infinitely many integer solution.*

Hint. Note that $x = 2m^2 + 1$ and $y = 2m$ is a solution. Now find recursively further solutions using the following argument: By the binomial theorem we have that, for each n $\in \mathbb{N}$,

$$\left(x + y\sqrt{D}\right)^n = x_n + y_n\sqrt{D} \text{ and } \left(x - y\sqrt{D}\right)^n = x_n - y_n\sqrt{D}$$

for some pair (x_n, y_n) of integers. Also

$$1 = \left(x^2 - Dy^2\right)^n = \left(x + y\sqrt{D}\right)^n\left(x - y\sqrt{D}\right)^n = x_n^2 - Dy_n^2 .$$

TRANSFORMATION METHODS

Svetoslav Bilchev, Emiliya Velikova
UNION OF BULGARIAN MATHEMATICIANS

Section 1. INTRODUCTION

The world famous mathematician COURANT has defined the mathematician as "a man who works very hard at being lazy". As we think he had wanted to illustrate one of the most important qualities of every serious mathematician, that was called "work-laziness duality", which is the base of the transform theory.

Although the transform theory is usually though like some certain complicated definite integrals, we prefer to take a broader view, which will give to us the possibility to see just what it is that goes "on behind the scenes".

But actually which is this simultaneously laziness and very hard - working mathematician?

Here it is the base problem from the view of our laziness-serious worker. Really, very often the mathematician is faced with a difficult problem which he certainly must solve. But, like a "lazy", he wishes the problem to be an easy problem or at least an easier problem. And being a good and very hard worker, he endeavors to find the way for reducing it to the same.

The action in which the hard problem reduces itself to an easy (or an easier) problem names a transformation and the corresponding method names the transform method.

Hence, the mathematician who wants to go from the box on the left to the box on the right in Fig. 1 and finding himself unable (or "too lazy") to do it so directly, he prefers to take the circuitous and may be longest, at the first view, path in Fig. 2.

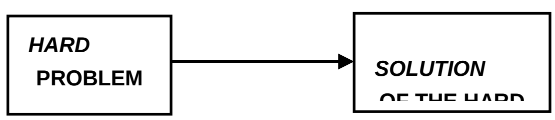


Figure 1

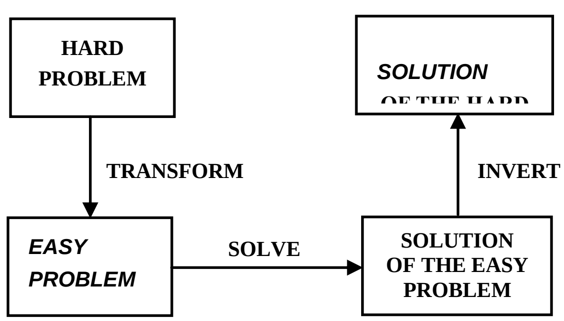


Figure 2

It turned out that this circuitous path is exceptionally intellectually concentrated and astonishingly effective not only in the high mathematics but also in the classic (elementary) mathematics.

Section 2. SOME TRANSFORMATIONS

2.1. Transformation for numbers

In ancient times, the multiplication of two numbers was something practiced only by the geniuses of the day. No wonder either, the multiplication being carried out in Roman numerals. Today a grade - school boy can easily knock off

(LXVII) times (XXXIV) .

Problem 2.1.1. Calculate the product (LXVII) . (XXXIV)

Solution. The main idea is to transform the Roman numbers into the ordinary digital system.

***First step**:* (LXVII) . (XXXIV) = 67**.**34

Second step

$$\begin{array}{r} 67 \\ \times\, 34 \\ \hline 268 \\ +\,201 \\ \hline 2278 \end{array}$$

***Third step**:* 2278 = MMCCLXXVIII .

Thus (LXVII) **.** (XXXIV) = MMCCLXXVIII .

This school boy (who obviously multiplies better then we do) has already grasped the whole philosophy of transform theory. He took the three steps:

1. ***First step:*** **Transform**
2. ***Second step:*** **Solve**
3. ***Third step:*** **Invert**

in a very clearly delineated way.

Even this simply little example brings out one of the basic properties of a transform. A transform corresponds to a representation.

2.2. Transformations for sums

Problem 2.2.1. Calculate the sum:

(1) $$\sum_{k=1}^{n} \frac{k}{k^4 + k^2 + 1} .$$

Solution. The main idea is to “zoom” the sum. Let $F(k) = \dfrac{1}{k^2 + k + 1}$. Then by the equality

$$\frac{k}{k^4 + k^2 + 1} = \frac{1}{2}\left[F(k-1) - F(k)\right]$$

the sum (1) can be presented as follows

(2) $$\frac{1}{2}\left(\left[F(0)-F(1)\right]+\left[F(1)-F(2)\right]+\cdots+\left[F(n-1)-F(n)\right]\right)=.$$
$$=\frac{1}{2}\left[F(0)-F(n)\right].$$

Then we get from (2) the answer of (1): $\dfrac{n^2+n}{2\left(n^2+n+1\right)}$.

Problem 2.2.2. Calculate the sum

(3) $$\sum_{k=2}^{n}\frac{1}{(k-1)k(k+1)}.$$

Solution. It is obvious that

(4) $$\sum_{k=2}^{n}\left[F(k-1)-2F(k)+F(k+1)\right]=F(1)-F(2)-F(n)+F(n+1) \quad .$$

Let we put $F(k)=\dfrac{1}{k}$ in (4). Then the sum (3) is equal to $\dfrac{(n-1)(n+2)}{4n(n+1)}$.

2.3. Transformations with applying of the basic triangle inequality

Below we will use so named **"basic triangle inequality"** for the triangle ABC, i.e.

$$AB+BC\geq AC \quad .$$

Problem 2.3.1. Find the best low bound of the function:

$f(x)=\sqrt{x^2-x+1}+\sqrt{x^2-x\sqrt{3}+1}$, where x is real variable.

Solution. The main idea is to present the expressions $\sqrt{x^2-x+1}$ and $\sqrt{x^2-x\sqrt{3}+1}$ as sides of a triangle and to apply the basic triangle inequality.

<u>*Case 1*</u>**.** Let <u>*x > 0*</u>. For ΔABC with $\angle ACB=90^0$, $AC=BC=1$ we construct a line through the point *C* which divide $\angle ACB$ into two angles, respectively, equal to 30^0 and 60^0 (Fig.3). On this line we take a point *M* such that $CM=x>0$. Then $AB=\sqrt{1^2+1^2}=\sqrt{2}$.

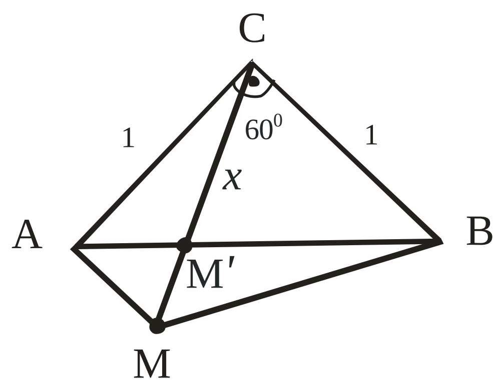


Figure 3

By the Cosine theorem we calculate

$$AM=\sqrt{1^2+x^2-2.1.x\,cos\,30^\circ}=\sqrt{x^2-\sqrt{3}x+1}, \qquad (\Delta AMC)$$

and

$$BM=\sqrt{1^2+x^2-2.1.x\,cos\,60^\circ}=\sqrt{x^2-x+1}, \qquad (\Delta BMC).$$

Thus $f(x)=AM+BM$ and the problem can be transformed to find the best low value of the broken line AMB. Thus $f(x)=AM+MB\geq AB$, i.e.

(5) $$\sqrt{x^2-x\sqrt{3}+1}+\sqrt{x^2-x+1}+\geq\sqrt{2}\Leftrightarrow f(x)\geq\sqrt{2} \quad .$$

The equality occurs in (5) when $M\equiv M'$ or $x=CM'$. In this case we calculate that: $\dfrac{x}{\sin 45^o}=\dfrac{1}{\sin 75^o}$. It follows that $x=\sqrt{3}-1$ (Fig. 4).

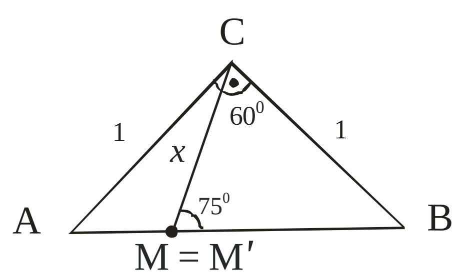


Figure 4

Hence $f\left(\sqrt{3}-1\right)=\sqrt{2}$*, i.e. the best low bound of* $f(x)$ *is equal to* $\sqrt{2}$.

<u>*Case 2*</u>*.* If <u>*x = 0*</u>, it is obvious that $f(0)=1+1>\sqrt{2}$.

<u>*Case 3.*</u> Let <u>*x < 0*</u>. Analogically, we construct the point M' on the line CM' (on the other site of the point *C*), such that $CM'=-x>0$ (Fig. 5). Then $AM'+BM'\geq\sqrt{2}$, i.e. (5).

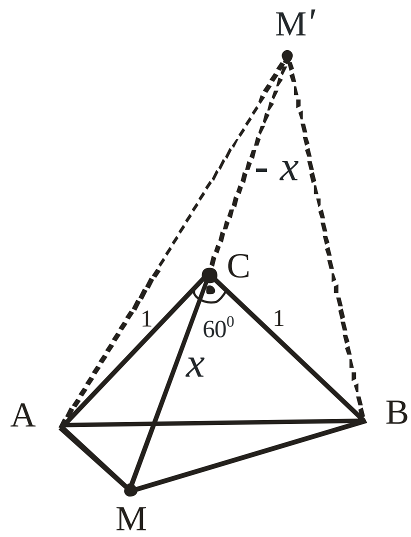


Figure 5

Problem 2.3.2. Solve the inequality

(6) $$\sqrt{35x^2+6\sqrt{6}x+2}+\sqrt{35x^2-8\sqrt{6}x+3}\geq\frac{7}{\sqrt{5}}.$$

Solution. We apply the same idea for transforming the expressions $\sqrt{35x^2+6\sqrt{6}x+2}$, $\sqrt{35x^2-8\sqrt{6}x+3}$, $\frac{7}{\sqrt{5}}$ into sides of a triangle. So we construct the quadrangles $ACBM$
(case $x>0$) or $ACBM'$ (case $x<0$) and we apply $AM+BM\geq AB$ or $AM'+BM'\geq AB$.

<u>*Case 1.*</u> Let <u>*x > 0*</u> and $\cos\alpha=-\frac{3\sqrt{3}}{\sqrt{35}}$, $\cos\beta=\frac{4\sqrt{2}}{\sqrt{35}}$ (Fig. 6). Then

(7) $$AM+BM=\sqrt{35x^2+6\sqrt{6}x+2}+\sqrt{35x^2-8\sqrt{6}x+3}\geq AB=\frac{7}{\sqrt{5}}.$$

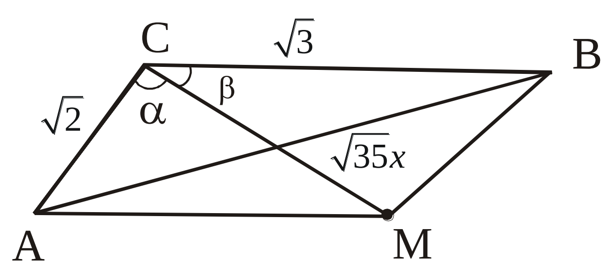


Figure 6

<u>*Case 2.*</u> Let <u>*x < 0.*</u> We transform $\left(\sqrt{35}x,\alpha,\beta\right)\rightarrow\left(-\sqrt{35}x,\pi-\alpha,\pi-\beta\right)$, i.e. $M\rightarrow M'$ (Fig. 7). Analogically, (7) is true for the point M', i.e. (6) is true when <u>*x < 0*</u>.

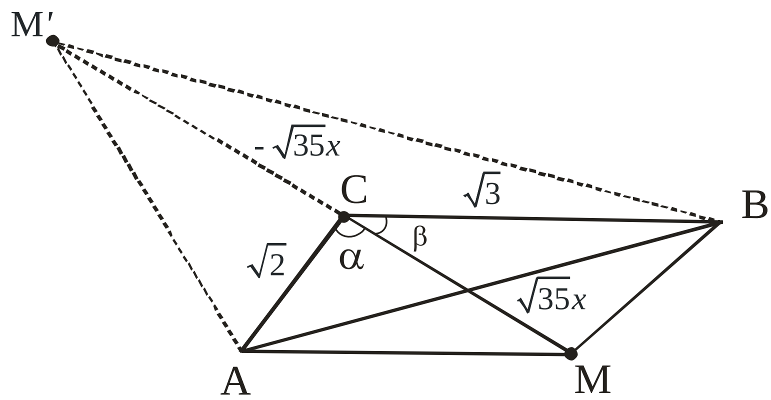


Figure 7

<u>Case 3.</u> Let <u>*x = 0*</u>. The inequality (6) is true because $\sqrt{2}+\sqrt{3}>\frac{7}{\sqrt{5}}$.

Hence, the inequality (6) is true for every real x.

Problem 2.3.3. If x, y, z are arbitrary positive numbers, prove the inequality

(8) $$\sqrt{x^2-xy+y^2}+\sqrt{y^2-yz+z^2}\geq\sqrt{x^2+xz+z^2}\ .$$

Solution. We apply the same idea as before but the construction of the triangle is new one.

Let we construct the quadrangle $ABCD$ (Fig. 8), where $DA=x>0,\ DB=y>0,$ $DC=z>0,\ \angle ADB=\angle BDC=60^{\circ}$.

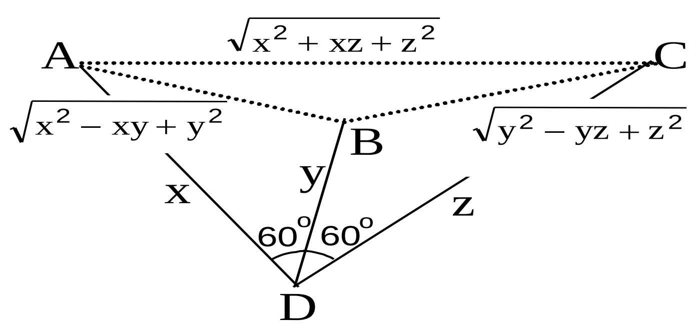


Figure 8

By applying the Cosine theorem for the triangles ΔADB, ΔBDC, ΔADC respectively we find that

(9) $$AB=\sqrt{x^2-xy+y^2},\ BC=\sqrt{y^2-yz+z^2},\ AC=\sqrt{x^2+xz+z^2}\ .$$

The equality (8) immediately follows from the existence of ΔABC, i.e. from (9) and $AB+BC\geq AC$.

The equality in (8) occurs only when the point $B\in AC$. Then

$F_{\Delta ADB} + F_{\Delta BDC} = F_{\Delta ADC}$, i.e.

$$\frac{1}{2}xy\sin 60^0 + \frac{1}{2}yz\sin 60^0 = \frac{1}{2}zx\sin 120^0 \Leftrightarrow xy + yz = xz \Leftrightarrow \frac{1}{x} + \frac{1}{z} = \frac{1}{y}.$$

2.4. Transforming of algebraic inequalities into geometric inequalities

Problem 2.4.1. For $x, y, z > 0$ prove the inequality

(10) $$\sqrt{y^2 + yz + z^2} + \sqrt{z^2 + zx + x^2} + \sqrt{x^2 + xy + y^2} \geq 3\sqrt{yz + zx + xy} .$$

Solution. It is obvious from the given construction (Fig. 9) that the inequality (10) is equivalent to the following well-known geometric inequality $a + b + c \geq 2\sqrt{3\sqrt{3}F}$, i.e. $a + b + c \geq 6r\sqrt{3}$.

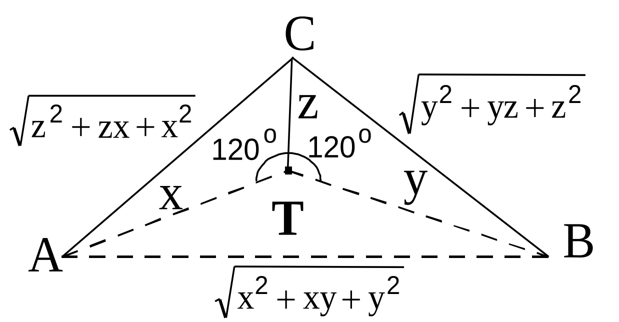


Figure 9

Problem 2.4.2. For $x, y, z > 0$ prove the inequality

(11) $$\frac{x + y + z}{3\sqrt{3}} \geq \frac{yz + zx + xy}{\sqrt{x^2 + xy + y^2} + \sqrt{y^2 + yz + z^2} + \sqrt{z^2 + zx + x^2}}.$$

Solution. We use the same construction (Fig. 9) for proving that (11) is equivalent to the inequality $x + y + z = AT + BT + CT \geq 2\sqrt{F\sqrt{3}}$. The inequality is true because it is a special case of the well-know inequality $R_1 + R_2 + R_3 \geq 2\sqrt{F\sqrt{3}}$ (Fig. 10) which is done for every internal point *P*.

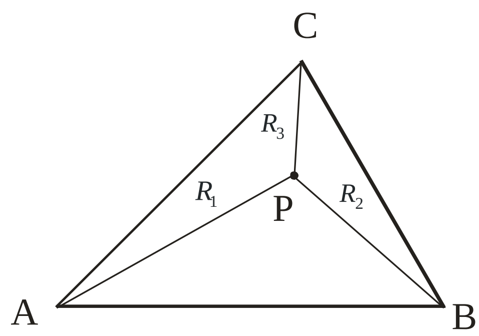


Figure 10

Problem 2.4.3. For $x,y,z>0$ prove the inequality

(12) $$\left(\sum x\right)^2\left(\sum yz\right)^2 \le 3\prod\left(y^2+yz+z^2\right).$$

Solution. We find that (12) is equivalent to the inequality

(13) $$AT+BT+CT \le 3R$$

by applying the same transformation (Fig. 9).
It is well-known that the Torichelly's point *T* has the following property (Fig 10)

(14) $$x+y+z=TA+TB+TC \le PA+PB+PC.$$

Then it is necessary to choose $P\equiv O$, where *O* is the circum centre of ΔABC, and (14) become (13), i.e. (12).

2.5. Transformations for algebraic problems

Problem 2.5.1. Find the best low bound of the function

$$f(x,y)=\frac{x^2+3y^2}{\sqrt{\left(7x^2+3y^2\right)\left(x^2+12y^2\right)}},\ \text{where } (x,y)\neq(0,0).$$

Solution. At first, we have to make a very important observation that

$$9\left(x^2+3y^2\right)=2\left(x^2+12y^2\right)+\left(7x^2+3y^2\right).$$

Hence, if apply the algebraic transformation

$$2\left(x^2+12y^2\right)=u>0,\quad i.e.\quad x^2+12y^2=\frac{u}{2}$$ and

$7x^2+3y^2=v>0$, then

$$x^2+3y^2=\frac{u+v}{9} \quad \text{and} \quad f(x,y)\equiv g(u,v)=\frac{\frac{u+v}{9}}{\sqrt{v\cdot\frac{u}{2}}}=\frac{\sqrt{2}}{9}\cdot\frac{u+v}{\sqrt{uv}} \; .$$

But from well - know inequality: $u+v\geq 2\sqrt{uv}$, i.e. $\frac{u+v}{\sqrt{uv}}\geq 2$ when $u,v>0$, it follows

that $$f(x,y)\equiv g(u,v)=\frac{\sqrt{2}}{9}\cdot\frac{u+v}{\sqrt{uv}}\geq\frac{\sqrt{2}}{9}\cdot 2=\frac{2\sqrt{2}}{9} \quad \text{or} \quad f(x,y)\geq\frac{2\sqrt{2}}{9}.$$

The equality occurs if $u=v$, i.e.

$$2\left(x^2+12y^2\right)=7x^2+3y^2 \qquad and \qquad x:y=\pm\sqrt{21}:\sqrt{5} \; .$$

Problem 2.5.2. Let x,y,z are real numbers, such that

(15) $$x^2+2y^2+z^2=\frac{2}{5}a^2, \;\; a>0.$$

Find the best low and upper bounds of the expression $x-y+z$.

Solution. Let $u=x-y+z$. Then $x=u+y-z$ and (15) transforms itself into:

$$(u+y-z)^2+2y^2+z^2-\frac{2}{5}a^2=0 \quad \text{, i.e.}$$

(16) $$3y^2+2(u-z)y+2z^2-2uz+u^2-\frac{2}{5}a^2=0.$$

The equality (16) has to have only real roots with respect to y, i.e.

$$D_1=(u-z)^2-3\left(2z^2-2uz+u^2-\frac{2}{5}a^2\right)\geq 0 \text{ or } 5z^2-4uz+2u^2-\frac{6}{5}a^2\leq 0.$$

If we suppose that

(17) $$D_2=(2u)^2-5\left(2u^2-\frac{6}{5}a^2\right)=6a^2-6u^2<0,$$

then the inequality (16) has not any real solution for *z*. Hence

$D_2=6a^2-6u^2\geq 0$, then $u^2\leq a^2$ or $-a\leq u\leq a$, i.e: $-a\leq x-y+z\leq a$.

2.6. Transformations for algebraic systems

The main idea in this section is to transform the given algebraic problems into geometric ones.

Problem 2.6.1. Let $a,b,c>0$ for which

1. (18) $$\sqrt{a}+\sqrt{b}+\sqrt{c}=\frac{\sqrt{3}}{2}.$$

2. Prove that the system

(19) $$\begin{cases}\sqrt{y-a}+\sqrt{z-a}=1\\ \sqrt{z-b}+\sqrt{x-b}=1\\ \sqrt{x-c}+\sqrt{y-c}=1\end{cases}$$ has exactly one solution.

Solution. We construct an equilateral triangle *ABC* with unit sides. Then the altitude of this triangle is equal to $\frac{\sqrt{3}}{2}$. There exists just one point *M,* inside of the triangle, which distances to *BC* and *CA* are equal to $\sqrt{a}$ and $\sqrt{b}$ respectively. This point M is the only one intersection point of the lines a_1, b_1 which are parallel to *BC* and *CA* and have the distances $\sqrt{a}$ and $\sqrt{b}$ to them respectively (Fig. 11).

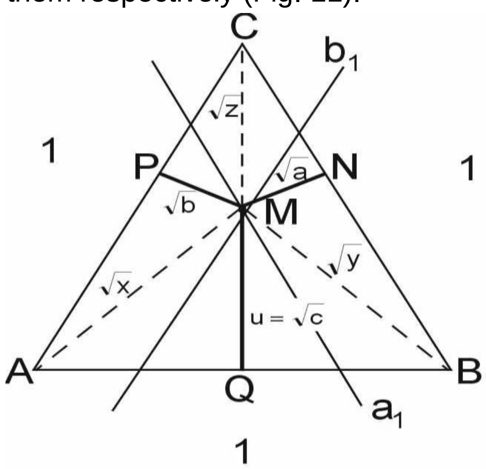


Figure 11

Let the last one distance from M to AB is equal to **u.** Then

$$F_{\Delta BMC}+F_{\Delta CMA}+F_{\Delta AMB}=F_{\Delta ABC}$$

and

$$\frac{1}{2}\sqrt{a}\cdot 1+\frac{1}{2}\sqrt{b}\cdot 1+\frac{1}{2}\cdot u\cdot 1=\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1$$, i.e.

(20) $$\sqrt{a}+\sqrt{b}+u=\frac{\sqrt{3}}{2}.$$

From (18) and (20) it follows that $u=\sqrt{c}$. Therefore, the distances *AM, BM, CM* are determined in only one way by the uniqueness of the constructed point *M* when *a, b, c > 0* are given, satisfying (18). They are exactly equal to $\sqrt{x}$, $\sqrt{y}$, $\sqrt{z}$ respectively as it follows from (19) and the Fig. 11.

Really, the system (19) is equivalent to the system

$$\begin{vmatrix} BN+NC=1 \\ CP+PA=1 \\ AQ+QB=1 \end{vmatrix}$$

and then it has exactly one solution

$$x=AM^2,\ \ y=BM^2,\ \ z=CM^2.$$

Problem 2.6.2. Solve the system

$$(21)\qquad \begin{vmatrix} x=\sqrt{y^2-a^2}+\sqrt{z^2-a^2} \\ y=\sqrt{z^2-b^2}+\sqrt{x^2-b^2} \\ z=\sqrt{x^2-c^2}+\sqrt{y^2-c^2} \end{vmatrix},\quad \text{where } a,b,c>0.$$

Solution. We construct a triangle with sides *x, y, z* and altitudes *a, b, c* respectively (Fig. 12).

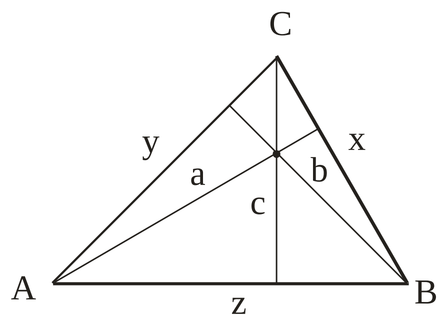


Figure 12

Then it is absolutely evident how to obtain the system (21).
Now we have to find the sides of so constructed triangle. From

$$ax=by=cz=2F=2\sqrt{s(s-a)(s-b)(s-c)}$$

it follows that then the solution of the system (21) is

$$x=\frac{2F}{a},y=\frac{2F}{b},z=\frac{2F}{c},\quad \text{where}$$

$$F=\left[\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\right]^{-\frac{1}{2}}.$$

2.7. Transformations for algebraic and trigonometric problems

Here we will use so named **“cyclic sums or cyclic products”**, i.e. for example:

$$\sum \frac{1}{yz} = \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} \quad \text{or} \quad \Pi tg\alpha = tg\alpha . tg\beta . tg\gamma \ .$$

Problem 2.7.1. If the angles $\alpha, \beta, \gamma \in \left(o, \frac{\pi}{2} \right)$ are such that

(22) $\qquad tg^2\alpha + tg^2\beta + tg^2\gamma = a$, $a \in (0,2]$,

prove the inequality

(23) $\qquad tg\gamma \le \dfrac{\cos(\alpha - \beta)}{\sin(\alpha + \beta)}$.

Solution. The main idea is to transform the trigonometric problem (22) and (23) into an algebraic inequality

(24) $\qquad z \le \dfrac{1 + xy}{x + y}$,

where $x,\ y,\ z$ are positive real numbers, such that $\sum x^2 = a$, $a \in (0,2]$.

From the obvious inequality $(x + y - z)^2 \ge 0$ consequently follows that

$$x^2 + y^2 + z^2 + 2(xy - xz - yz) \ge 0 \quad \Leftrightarrow$$

$$yz + zx - xy \le \frac{x^2 + y^2 + z^2}{2} = \frac{a}{2} \le 1 \quad \Rightarrow \quad z(x + y) \le xy + 1 \quad \Leftrightarrow$$

(24).

At the end we substitute $x = tg\alpha, y = tg\beta, z = tg\gamma$ in (24) and then we get

$$tg\gamma \le \frac{1 + tg\alpha \cdot tg\beta}{tg\alpha + tg\beta} = \frac{\cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta}{\sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta} = \frac{\cos(\alpha - \beta)}{\sin(\alpha + \beta)}.$$

Problem 2.7.2. Let $x, y, z > 0$ are such that

1. $\qquad x + y + z = xyz$.

Prove the equation

(26) $\qquad \sum \dfrac{1}{yz} \left[\sqrt{1 + y^2}\sqrt{1 + z^2} - \sqrt{1 + y^2} - \sqrt{1 + z^2} \right] = 0.$

Solution. We will use the following transformation:

$$x = tg\alpha > 0,\ y = tg\beta > 0,\ z = tg\gamma > 0 \quad \text{with} \quad \alpha, \beta, \gamma \in \left(0, \frac{\pi}{2} \right).$$

The equality

$$tg(\alpha+\beta+\gamma)=0\text{, i. e. } \alpha+\beta+\gamma=\pi$$

follows from the identity

$$tg(\alpha+\beta+\gamma)=\frac{\sum tg\alpha-\prod tg\alpha}{1-\sum tg\beta\, tg\gamma}$$

and from (25) .

Then it follows that

$$\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=\frac{\pi}{2} \text{ and } cot\, g\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}\right)=\frac{1-\sum tg\frac{\beta}{2}tg\frac{\gamma}{2}}{\sum tg\frac{\alpha}{2}-\prod tg\frac{\alpha}{2}}=0.$$

Hence,

1. $$\sum tg\frac{\beta}{2}tg\frac{\gamma}{2}=1.$$

But $tg\dfrac{\alpha}{2}=\dfrac{\sqrt{1+x^2}-1}{x}$, etc., and from (27) we obtain consequently:

2. $$\sum\frac{\sqrt{1+y^2}-1}{y}\cdot\frac{\sqrt{1+z^2}-1}{z}=1\ ,$$

3. $$\sum\frac{1}{yz}\left[\sqrt{1+y^2}\sqrt{1+z^2}-\sqrt{1+y^2}-\sqrt{1+z^2}+1\right]=1\ ,$$

4. $$\sum\frac{1}{yz}\left[\sqrt{1+y^2}\sqrt{1+z^2}-\sqrt{1+y^2}-\sqrt{1+z^2}\right]+\sum\frac{1}{yz}=1.$$

We obtain (26) from (30) and (25), because (25) is equivalent to $\sum\dfrac{1}{yz}=1$.

2.8. Transformations with vectors

Problem 2.8.1 . If *a, b, c, x, y, z* are arbitrary real numbers prove the inequality:

(31) $$\sum ax+\left(\sqrt{\sum a^2.\sum x^2}\right) \geq \frac{2}{3}\sum a.\sum x\ .$$

Solution. It is possible to transform (31) consequently to the following inequalities:

$$3\sum ax+3\sqrt{\sum a^2\cdot\sum x^2} \geq 2\sum a\cdot\sum x \quad\Leftrightarrow$$

$$3\sqrt{\sum a^2\cdot\sum x^2} \geq 2\sum ax+2\sum a(y+z)-3\sum ax \quad\Leftrightarrow$$

(32) $$3\sqrt{\sum a^2 \cdot \sum x^2} \ \geq \ \sum a(2y+2z-x).$$

By applying the well-known Cauchy–Bunyakowsky–Schwartz's inequality for the vectors

$\vec{m}(a,\ b,\ c)$ and $\vec{n}(2y+2z-x,\ 2z+2x-y,\ 2x+2y-z)$

we obtain the following inequality

(33) $$\sum a(2y+2z-x) \ \leq \ \sqrt{\sum a^2 \cdot \sum (2y+2z-x)^2}\ .$$

The inequality (33) is equivalent to (32) because

(34) $$\sum (2y+2z-x)^2 = 9\sum x^2.$$

Really

$$\sum (2y+2z-x)^2 - 9\sum x^2 = \sum \left[(2y+2z-x)^2 - (3x)^2 \right] =$$

$$= \sum \left[(2y+2z-x)+3x \right]\left[(2y+2z-x)-3x \right] =$$

$$= \sum 2(x+y+z)2(y+z-2x) = 4(x+y+z)\sum (y+z-2x) = 0\ .$$

Then (32), i.e. (31), follows from (33) and (34).

Problem 2.8.2. Find all real solutions of the equation

(35) $$\sqrt{5\left(x^2+2yz\right)} + \sqrt{6\left(y^2+2zx\right)} + \sqrt{5\left(z^2+2xy\right)} = 4\sum x\,.$$

Solution. We transform the equation (35) by vectors

$\vec{b}\left(\sqrt{x^2+2yz}, \sqrt{y^2+2zx}, \sqrt{z^2+2xy} \right)$ and $\vec{a}\left(\sqrt{5}, \sqrt{6}, \sqrt{5}\right)$

into the vector equation: $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$, which is true only when $cos\left(\widehat{\vec{a},\vec{b}} \right) = 1$. Then the vectors $\vec{a}, \vec{b}$ have to be collinear, i.e.

(36) $$\frac{\sqrt{x^2+2yz}}{\sqrt{5}} = \frac{\sqrt{y^2+2zx}}{\sqrt{6}} = \frac{\sqrt{z^2+2xy}}{\sqrt{5}} \quad \text{or}$$

(37) $$x=a, y=2a, z=a \ \ \textit{and} \ \ x=5a, y=2a, z=5a\,,$$

where ***a*** is an arbitrary nonnegative number.

2.9. Transformations for sequences

Problem 2.9.1. The sequence $\{a_n\}$ is given by the conditions

(38) $$a_1 = 1,\ a_{n+1} = 2a_n + \sqrt{3a_n^2+1},\ n = 1,2,3,\ldots\ .$$

Find all elements of (38) which are natural numbers.

Solution. By the help of (38) we get $a_2 = 4$ and $a_n > 0$ for every natural number n. From $(a_{n+1} - 2a_n)^2 = 3a_n^2 + 1$ we obtain

(39) $\qquad a_n = 2a_{n+1} \pm \sqrt{3a_{n+1}^2 + 1}$.

If we have the sign „+“ in (39) then $a_n - 2a_{n+1} > 0$. But from (38) it follows $a_{n+1} - 2a_n > 0$, i.e. $a_n > 2a_{n+1} > 2.2a_n = 4a_n$. Thus $a_n < 0$ which is in contradiction with the proven at the beginning that $a_n > 0$ is done for every natural number n. Hence, from (39) we obtain

(40) $\qquad a_n = 2a_{n+1} - \sqrt{3a_{n+1}^2 + 1}$, $n = 1,2,3,\dots$.

Let we substitute *n* with $n-1$ in (40). Then we add the obtained equation $a_{n-1} = 2a_n - \sqrt{3a_n^2 + 1}$ to the recurrence condition in (38) and we get the following homogeneous recurrence equation:

(41) $\qquad a_{n+1} + a_{n-1} = 4a_n$.

The characteristic equation of (41) is $q^2 - 4q + 1 = 0$ with roots $q_{1,2} = 2 \pm \sqrt{3}$. Then

(42) $\qquad a_n = c_1\left(2+\sqrt{3}\right)^n + c_2\left(2-\sqrt{3}\right)^n$, $n = 1,2,3,\dots$,

where the constants c_1 and c_2 is necessary to get from the initial conditions: $a_1 = 1, \quad a_2 = 4$, i.e. $c_1 = -c_2 = \dfrac{1}{2\sqrt{3}}$. Then (42) becomes

$$a_n = \frac{1}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^n - \left(2-\sqrt{3}\right)^n\right],\ n = 1,2,3,\dots ,$$

which shows that a_n is natural number for every natural *n.*

Section 3. TRANSFORMATIONS FOR A TRIANGLE

3.1. T_ℓ transformation

Problem 3.1.1. For any triangle Δ with sides *a, b, c* and area *F* prove the Finsler-Hadwiger's inequality

(43) $\qquad 4F\sqrt{3} + \sum (b-c)^2 \le \sum a^2$.

Solution. We will use the following statement:

If there exists a triangle Δ with sides a, b, c and area F, then there exists a triangle Δ_1 with sides

$$\sqrt{a(s-a)}, \quad \sqrt{b(s-b)}, \quad \sqrt{c(s-c)}$$

and area $F_1 = \dfrac{F}{2}$.

Really, the triangle Δ_1 exists because

$$a(s-a)+b(s-b)-c(s-c)=2(s-a)(s-b)>0 \ ,$$

$$\left(\sqrt{a(s-a)}+\sqrt{b(s-b)}\right)^2 > a(s-a)+b(s-b) > \left(\sqrt{c(s-c)}\right)^2 \ ,$$

$$\sqrt{a(s-a)}+\sqrt{b(s-b)} > \sqrt{c(s-c)}.$$

The area F_1 of Δ_1 is equal to $\dfrac{F}{2}$, because from the Cosine theorem applied for the triangle Δ_1 it follows

$$cos\gamma_1 = \sqrt{\frac{(s-a)(s-b)}{ab}} \text{, i.e. } sin\gamma_1 = \sqrt{\frac{s(s-c)}{ab}} \text{ , or}$$

$$F_1 = \frac{1}{2}\sqrt{a(s-a)}.\sqrt{b(s-b)}.\sqrt{\frac{s(s-c)}{ab}} = \frac{F}{2} \ ,$$

where γ_1 is an angle of the triangle Δ_1. The statement is proved.

With the help of this statement we will formulate our main conclusion.

From any known geometric inequality

(44) $\qquad I(a,b,c,F) \geq 0$

it follows the following new (in general case) geometric inequality

(45) $\qquad I_{T_\ell} \equiv I\left(\sqrt{a(s-a)},\sqrt{b(s-b)},\sqrt{c(s-c)},\dfrac{F}{2}\right) \geq 0$.

The new inequality (45) we call **the T_ℓ-image of the initial inequality** *(44).*

And now is necessary only to find the T_ℓ-image of the well-known Weitzeböck's inequality

(46) $\qquad \sum a^2 \geq 4F\sqrt{3}$, i.e.

$$\sum\left(\sqrt{a(s-a)}\right)^2 \geq 2F\sqrt{3} \quad , \quad \sum a(s-a) \geq 2F\sqrt{3} \quad ,$$

(47) $\qquad 2\sum bc - \sum a^2 \geq 4F\sqrt{3}$,

which is obviously equivalent to the desired inequality (43).

It is not so difficult to see that the T_ℓ-image (43), i.e. (47), of the inequality (46) is better (sharper) inequality than the initial inequality (46).

Problem 3.1.2. If $s,R,r,h_a,h_b,h_c,r_a,r_b,r_c$ are the usual elements of any triangle, prove the inequalities

(48) $\qquad \dfrac{1}{2}\left(4R+r+s\sqrt{3}\right) \leq \sqrt{\dfrac{R}{2r}}.\sum\sqrt{h_a r_a} \leq \dfrac{1}{6}\left[5(4R+r)+s\sqrt{3}\right]$.

Solution. The double inequalities of Finsler-Hadwiger

(49) $$4F\sqrt{3}+\sum(b-c)^2 \le \sum a^2 \le 4F\sqrt{3}+3\sum(b-c)^2$$

are well-known (the left inequality in (49) is the inequality (43)) .
Now we will find the T_ℓ-image of the left inequality in (49), i.e. (47):

$$\sum 2\sqrt{bc(s-b)(s-c)}-\sum a(s-a)\ge 2F\sqrt{3} \quad ,$$

$$4\sum\sqrt{\frac{4Rrs}{a}.\frac{F^2}{s(s-a)}}\ge 4r^2+16Rr+4F\sqrt{3} \quad ,$$

(50) $$\sum\frac{1}{\sqrt{a(s-a)}}\ge\frac{4R+r+s\sqrt{3}}{2s\sqrt{Rr}} \quad ,$$

where we used the well-known identity: $2\sum a(s-a)=4r^2+16Rr$.

But

(51) $$\frac{1}{\sqrt{a(s-a)}}=\frac{\sqrt{h_a r_a}}{F\sqrt{2}}$$

and from (50) and (51) it follows

(52) $$\sum\sqrt{h_a r_a}\ge\frac{1}{2}\left(4R+r+s\sqrt{3}\right).\sqrt{\frac{2r}{R}} \; .$$

Further, analogically, we obtain the T_ℓ-image of the right inequality in (49):

(53) $$\sum\sqrt{h_a r_a}\le\frac{1}{6}\left[5(4R+r)+s\sqrt{3}\right].\sqrt{\frac{2r}{R}} \; .$$

The inequality (48) follows from (52) and (53).

3.2. Combination of transformations for a triangle

In this section we will give only an idea for using combinations of transformations.

Problem 3.2.1. Prove that the sides a,b,c and the area *F* of any triangle satisfy the inequality

(54) $$-3a^2+8b^2+12c^2\ge 24F \; .$$

When does the equality occur?

Solution. From the generalized inequality

(55) $$\left(1+\frac{2}{k}\right)a^2+(1+k)b^2-c^2\ge 4F$$

with equality if and only if

$$a : b : c = k : \sqrt{2} : \sqrt{k^2 + 2k + 2}$$

it follows the new concrete inequality ($k = 4$)

(56) $$3a^2 + 10b^2 - 2c^2 \geq 8F$$

with equality if and only if

$$a : b : c = 2\sqrt{2} : 1 : \sqrt{13}.$$

Now we will apply to the inequality (56) the combination of **the Parallelogram Transformation - PT(b)** with respect to the side *b* and **the Median Dual Transformation - MDT**. The formulas of the combination of those transformations are given below in the Table 1.

Table 1. Combination of transformations

Elements of the triangle Δ	**PT(b)**	**MDT**	**PT(b) ∗ MDT**
a	*a*	m_a	$\frac{1}{2}\sqrt{3a^2 - 2b^2 + 6c^2}$
b	$2m_b$	m_b	$\frac{1}{2}b$
c	*c*	m_c	$\frac{1}{2}\sqrt{6a^2 - 2b^2 + 3c^2}$
F	*F*	$\frac{3}{4}F$	$\frac{3}{4}F$

The inequality (54) is the final result of the combination of transformations: PT(b)∗MDT.
The equality occurs if and only if

$$\frac{1}{2}\sqrt{3a^2 - 2b^2 + 6c^2} : \frac{1}{2}b : \frac{1}{2}\sqrt{6a^2 - 2b^2 + 3c^2} = 2\sqrt{2} : 1 : \sqrt{13},$$

i.e. $$a : b : c = 2\sqrt{5} : 3 : \sqrt{5} \ .$$

COMPLEX NUMBERS IN GEOMETRY

Mircea Becheanu
ROMANIAN MATHEMATICAL SOCIETY

It is well known that there exists a bijective correspondence between the set of the points of a plane and the set C of complex numbers. This correspondence provides us with the advantage of using complex numbers in order to describe geometric situations. This description is possible not only because the set C is endowed with algebraic operations but also because it enjoys polar representation of complex numbers and fruitful functions like module and conjugation. We will see that using these elementary notions many geometric problems can have straightforward and nice solutions. So, complex numbers come out to be a wonderful world, rich in properties and easy to be handled. Our excursion in their land will be accomplished by a collection of problems which can be solved in a rather elegant way by using this tool.

In the forthcoming, if it is not stated otherwise, we will denote by capital letters A, B, C, ... , Z the points of the plane and by small letters a, b, c, ... , z their corresponding complex affixes. The complex affix a of a point A is also called the complex coordinate of that point. We start with some elementary but elegant examples and after that we will present some basic geometric configurations which can be successfully studied by using complex numbers.

1. Examples

1.1 Triangle inequality. Given complex numbers z_1, z_2 the following inequality is true

$$| z_1 + z_2 | \le | z_1| + |z_2|. \tag{1}$$

Proof. Since $|z|$ is a nonnegative real number, one may square the required inequality to obtain equivalent forms:

$$| z_1 + z_2 |^2 \le | z_1|^2 + 2|z_1||z_2| + |z_2|^2 \Leftrightarrow$$

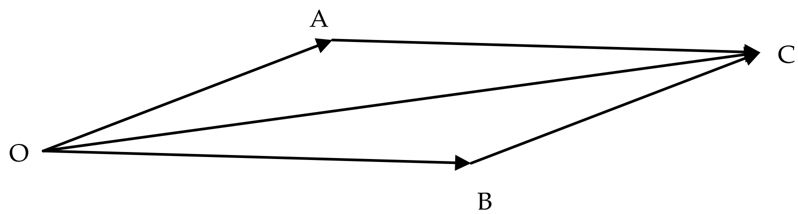
$$(z_1 + z_2)(\overline{z}_1 + \overline{z}_2) \le z_1\overline{z}_1 + z_2\overline{z}_2 + 2\sqrt{z_1\overline{z}_1 z_2\overline{z}_2} \Leftrightarrow$$

$$z_1\overline{z}_2 + z_2\overline{z}_1 \le 2\sqrt{z_1\overline{z}_1 z_2\overline{z}_2}\ .$$

Let us denote $z_1\overline{z}_2 = w$; then $\overline{w} = \overline{z}_1 z_2$. The last inequality becomes $w + \overline{w} \le 2|w|$. Set $w = x + iy$ and obtain that it is equivalent to $2x \le 2\sqrt{x^2 + y^2}$. But this is obvious.

When does the inequality (1) becomes an equality? It happens if and only if $x \ge 0$ and $y = 0$, which gives that $w = z_1\overline{z}_2 = \lambda$ is a nonnegative real number. After multiplying the above equality by z_2, one obtains $\lambda z_2 = |z_2|^2 z_1$. Since $\lambda \ge 0$, it follows that the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ which represent the numbers z_1, z_2 are collinear and similarly oriented.

The triangle inequality for complex numbers has a geometric interpretation: in a given triangle the sum of two sides is longer then the third side. Indeed, given the points A, B, C such that $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ one obtains the parallelogram OACB in which OA = |a|, AC = OB = |b|, OC = |a + b|. Since OC ≤ OA + AC, one has precisely |a + b| ≤ |a| + |b|.



1.2 Pompeiu's theorem.[3] Let A, B, C be an equilateral triangle and Z be a point on its plane, but not on its circumcircle. Then the line segments ZA, ZB, ZC are the sides of a triangle.

First solution. We start with a beautiful geometric solution. Assume that Z is an interior point of ΔABC (see Fig. 1). Draw through Z lines parallel to the sides AB, BC, CA which intersect these sides at the points P, M and N, respectively. The quadrilaterals APZN, PZMB and MZNC are isosceles trapeziums. Therefore one has ZA = PN, ZB = PM, ZC = MN and ΔMNP is the required triangle. A similar argument works when Z lies on a side of the triangle.

The second case is when Z is an exterior point. Consider the rotation about A by an angle of 60^0 which maps B onto the point C and let W be the image of Z under this rotation. Since rotations preserve distances we have ZC = WB and it is obvious that ZA = WA (Fig. 2). Therefore ΔZWC is the required triangle.

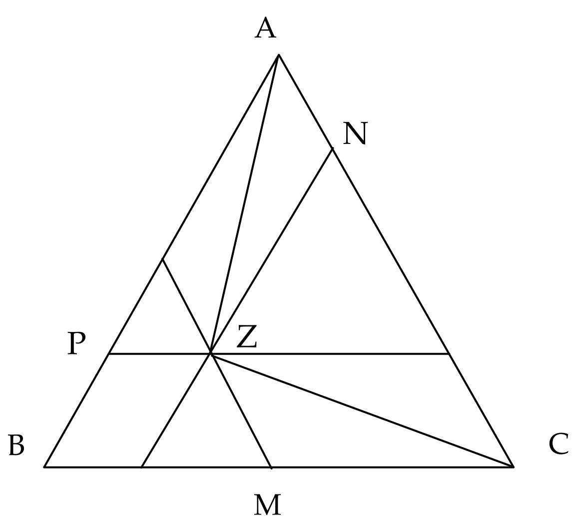


Figure 1

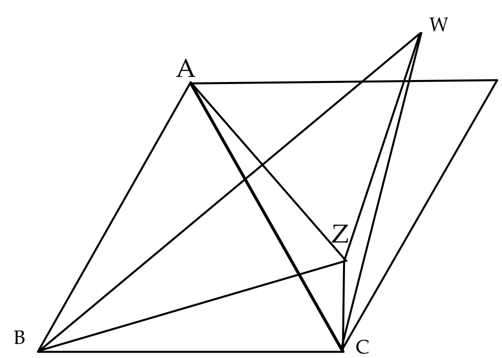


Figure 2

[3] Dimitrie Pompeiu (1873-1954) was a famous Romanian mathematician who was also interested in elementary mathematics.

We remark that the last argument works independently of the position of the point W, externally of ΔABC, or even inside it.

Second solution. When the solver has no intuition to draw parallel lines or to use rotations, he can refer to complex numbers. As usually suppose that the points A, B, C, Z have complex coordinates a, b, c, z. It is easy to see that the following identity holds:

$$(z - a)(b - c) + (z - b)(c - a) + (z - c)(a - b) = 0.$$

We may write it in the form

$$-(z - a)(b - c) = (z - b)(c - a) + (z - c)(a - b).$$

Then take modulus in the equality and apply the triangle inequality to obtain

$$|z - a||b - c| \le |z - b||c - a| + |z - c||a - b|.$$

Taking into account that $|a - b| = |b - c| = |c - a|$ one obtains

$$|z - a| \le |z - b| + |z - c|.$$

By the comment at the end of 1.1 it is easy to see that the last inequality cannot be an equality. This proves that the segments ZA, ZB, ZC are the sides of a triangle.

1.3. Ptolemy's theorem. In any convex quadrilateral ABCD the following inequality holds:

$$AC \cdot BD \le AB \cdot CD + AD \cdot BC.$$

This result is well known and it is usually proved by geometric arguments. Here we will give a short proof inspired by the previous problem.

Let a, b, c, d be the complex coordinates of the vertices A, B, C, D, respectively. As above, the following identity is true:

$$(a - b)(c - d) + (a - c)(d - b) + (a - d)(b - c) = 0.$$

After writing it in the form

$$(a - c)(b - d) = (a - b)(c - d) + (a - d)(b - c)$$

and then by applying the triangle inequality, one obtains

$$|a - c||b - d| \le |a - b||c - d| + |a - d||b - c|,$$

which is exactly the required result.

2. Lines and collinear points

2.1. Equation of a straight line. The points A, B, C are on a straight line if and only if the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear, i.e. $\overrightarrow{AB} = \lambda \overrightarrow{AC}$ for some real number λ. In terms of complex numbers we have either

$$\frac{b-a}{c-a} \in \mathbf{R},$$

or

$$\frac{a-b}{\overline{a}-\overline{b}} = \frac{a-c}{\overline{a}-\overline{c}}.$$

Therefore, given distinct points A and B, an arbitrary point Z belongs to the line AB if and only if

$$\frac{z-a}{\overline{z}-\overline{a}}=\frac{b-a}{\overline{b}-\overline{a}}.$$

This is the equation of a line in complex coordinates. It is convenient to write it in the form

$$z-a=\frac{b-a}{\overline{b}-\overline{a}}\left(\overline{z}-\overline{a}\right). \qquad (1)$$

Using the algebraic development of a determinant, it is easy to see that the line equation is equivalent to

$$\begin{vmatrix} z & \overline{z} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix}=0.$$

The ratio $\chi = \dfrac{b-a}{\overline{b}-\overline{a}}$ which appears in formula (1) is called the *complex slope* of the line AB. We remark that it is symmetric in the variables a, b and since $|b - a| = |\overline{b}-\overline{a}|$ it follows that $|\chi| = 1$. Therefore χ belongs to the unit circle and it can be expressed as $\chi = \cos\varphi + i\sin\varphi$. It is then convenient to write the equation (1) under the form

$$z-a=\chi\left(\overline{z}-\overline{a}\right). \qquad (2)$$

The geometric interpretation of the angle φ can be obtained from the definition of the slope. From the equality $b - a = \chi\left(\overline{b}-\overline{a}\right)$ and by considering the arguments of the numbers that appear in it, one obtains either one of the situations:

(i) $\dfrac{\varphi}{2} = \arg(b - a)$ when $0 \le \arg(b - a) < \pi$, or

(ii) $\dfrac{\varphi}{2} = \arg(b - a) - \pi$ when $\pi \le \arg(b - a) < 2\pi$.

Let us assume that the point Z divides the segment AB into a given ratio $AZ : ZB = \lambda$, $\lambda \in R$, $\lambda \neq -1$. Then one obtains from (1) the exact expression for z:

$$z = \frac{1}{1+\lambda}a + \frac{\lambda}{1+\lambda}b. \qquad (3)$$

We say that the above equation represents the *parametric equation* of the line through the points A, B. We will understand it in the following way: when λ runs through the set **R** of real numbers, except $\lambda = -1$, Z runs through the points of the line AB. In particular, when $\lambda = 1$, Z is the midpoint of AB and

$$z = \frac{1}{2}(a+b).$$

Also, when λ takes all real positive values, Z runs over the point segment AB and we say that Z is a convex combination of A and B.

2.2 The area of a triangle. Given a triangle ABC one may compute its area in terms of complex coordinates of its vertices. It is given by the formula [ABC] = $\frac{1}{4i}|\Delta|$ where

$$\Delta = \begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}.$$

For the proof, one may use the formula which gives the area in terms of the coordinates of the points in a plane: if $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$ then [ABC] = $\frac{1}{2}|\Delta|$ where

$$\Delta = \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$

Hence it is sufficient to show that

$$\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix} = \frac{1}{2i}\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$

This computation is straightforward.

As an application of the previous considerations we will present a simple solution of an IMO problem.

2.3. Problem.[4] The diagonals AC and CE of a regular hexagon ABCDEF are divided by the internal points M, N respectively, such that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Find the values of r if B, M and N are collinear.

Solution. Let $\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ be the complex cubic root of unity. Since the algebra of this number is well known we assume that the affixes of the vertices A, B, ... , F are the complex numbers (see Fig.1)

$a = 1$, $b = 1 + \varepsilon$, $c = \varepsilon$, $d = -1$, $e = \varepsilon^2$, $f = 1 + \varepsilon^2$.

[4] Problem 5, from IMO 1982

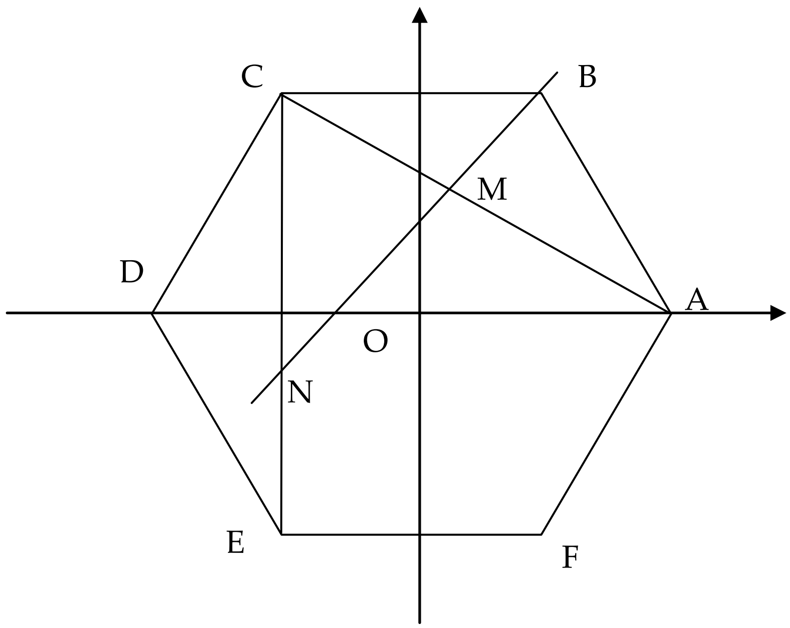


Figure 1

The condition of the problem can be translated into a condition about the complex coordinates of the points M, N (see (2.1)) as follows:

$$\frac{m-a}{c-a}=\frac{n-c}{e-c}=r\,,\ \ r\in \mathbf{R},\ \ 0<r<1.$$

From the above, one obtains the expressions

$$m = 1 + r(\varepsilon - 1) = (1 - r) + r\varepsilon$$

and

$$n = \varepsilon m = (1 - r)\varepsilon + r\varepsilon^2 = -r + (1 - 2r)\varepsilon.$$

The condition of collinearity gives

$$\begin{vmatrix} 1+\varepsilon & 1+\varepsilon^2 & 1 \\ m & \overline{m} & 1 \\ \varepsilon m & \varepsilon^2\overline{m} & 1 \end{vmatrix}=0.$$

Looking at the expression of m we obtain an equation in r and the problem reduces to solving this equation. It is important to handle the algebraic computations in such a way that one obtains a convenient final form:

$$\begin{vmatrix} 1+\varepsilon & 1+\varepsilon^2 & 1 \\ m & \overline{m} & 1 \\ \varepsilon m & \varepsilon^2\overline{m} & 1 \end{vmatrix}=\begin{vmatrix} -\varepsilon^2 & -\varepsilon & 1 \\ m & \overline{m} & 1 \\ \varepsilon m & \varepsilon^2\overline{m} & 1 \end{vmatrix}=(m\overline{m}-m-\overline{m})(\varepsilon^2-\varepsilon)=0.$$

This equation is successively equivalent to

$$m\overline{m}-m-\overline{m}=0 \Leftrightarrow (m-1)(\overline{m}-1)=1 \Leftrightarrow |m-1|^2=1 \Leftrightarrow |r(\varepsilon-1)|^2=1 \Leftrightarrow 3r^2=1.$$

Given the restrictions on r, the required solution is $r = \frac{1}{\sqrt{3}}$.

3. Angles

3.1. The angle between two lines. Similar triangles. Given complex numbers a, b and A, B their corresponding points, the angle $\angle AOB$ is given by the formula

$$\angle AOB = \arg \frac{b}{a} = \arg b - \arg a.$$

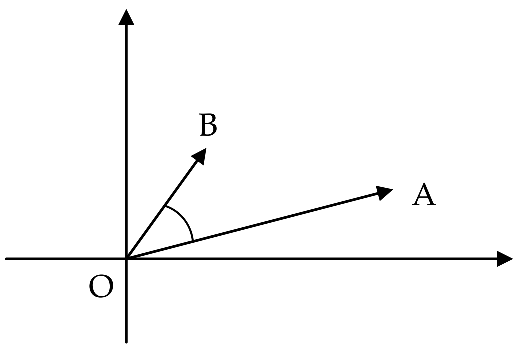


Figure 1

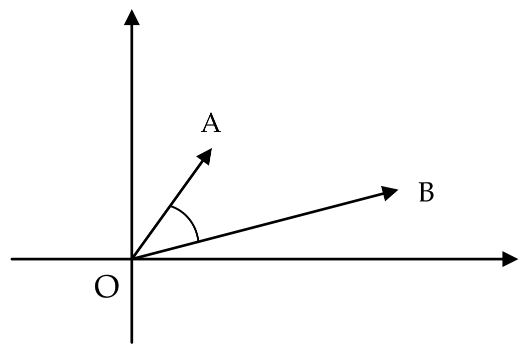


Figure 2

Given three points A, B, C one has

$$\angle ABC = \arg \frac{a - b}{c - b}.$$

These considerations can be applied in various situations. For example, given four distinct points A, B, C, D, the lines AB and CD are perpendicular if and only if the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular. This is equivalent to

$$\frac{b - a}{d - c} = \lambda i \qquad \text{where } \lambda \in \mathbf{R}, \lambda \neq 0.$$

The condition is equivalent to the following equality:

$$\frac{b - a}{d - c} = - \frac{\overline{b} - \overline{a}}{\overline{d} - \overline{c}}.$$

Another example is the characterisation of *similar triangles*. Let $A_1A_2A_3$ and $B_1B_2B_3$ be two triangles and let assume that they have the same orientation. The triangles are similar if and only if

$\frac{A_1A_2}{A_1A_3} = \frac{B_1B_2}{B_1B_3}$ and $\angle A_2A_1A_3 = \angle B_2B_1B_3$. Using complex numbers, these conditions are equivalent to the following:

$$\frac{a_2 - a_1}{a_3 - a_1} = \frac{b_2 - b_1}{b_3 - b_1}.$$

Using the expanding formula of a determinant, the above condition can be written in the form

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

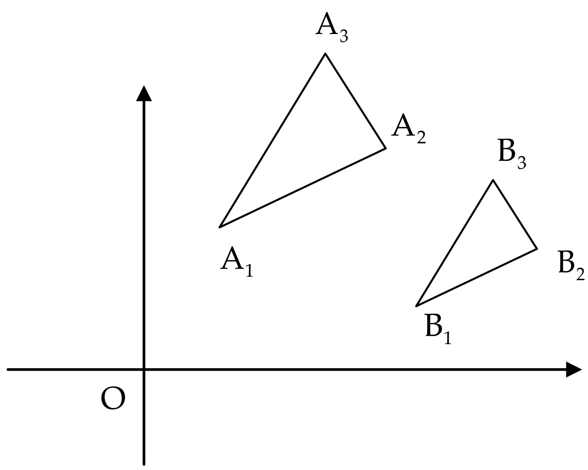


Figure 6

In the case when the triangles $A_1A_2A_3$ and $B_1B_2B_3$ have different orientations then one may take the triangle $B_1^{'}B_2^{'}B_3^{'}$ whose vertices are the reflexions of B_1, B_2, B_3 along the axis Ox. It has the same orientation as $\Delta A_1A_2A_3$. The complex coordinates of its vertices are the conjugate numbers $\overline{b}_1$, $\overline{b}_2$, $\overline{b}_3$. Then the condition of similarity is

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ \overline{b}_1 & \overline{b}_2 & \overline{b}_3 \end{vmatrix} = 0.$$

3.2. Rotations. The equilateral triangle. Given a point Z_0 and an angle φ, where $0 \le \varphi < 2\pi$, we define the rotation about Z_0 by angle φ to be the function defined on the complex plane which maps an arbitrary point Z to the point Z' such that $ZZ_0 = Z'Z_0$ and $\angle ZZ_0Z' = \varphi$. Therefore, by the above considerations we have $\frac{z' - z_0}{z - z_0} = \cos\varphi + i\sin\varphi$. Set $\omega = \cos\varphi + i\sin\varphi$. Then z' is given by the formula

$$z' = \omega z + (1 - \omega)z_0.$$

It is called the *analytic formula of a rotation*. Using it, one can easily prove many properties of rotations. For example, we will show that any rotation preserves distances.

Let w be the complex coordinate of a second point and let w' be its image by the same rotation of angle φ about z_0. Then $w' = \omega w + (1 - \omega)z_0$. A standard computation shows that

$$|z' - w'| = |\omega z + (1 - \omega)z_0 - \omega w - (1 - \omega) z_0| = |\omega||z - w| = |z - w|.$$

As an application of the rotation formula we will give the condition for the vertices of an equilateral triangle: three distinct points A, B, C are the vertices of an equilateral triangle if and only if their affixes a, b, c satisfy the equality

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 0.$$

Proof. The triangle ABC has in the plane either a counter-clockwise (see Fig.1) or a clockwise (see Fig. 2) orientation.

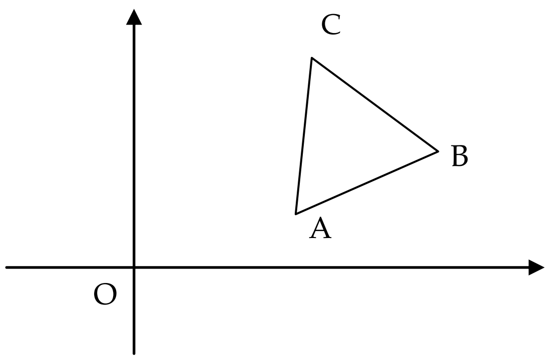


Fig.1

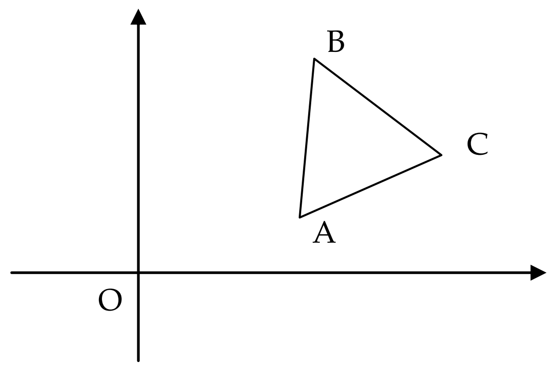


Fig.2

In the first case, C is obtained from B by a rotation of 60^0 about A. Hence the rotation is given by multiplication by $\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = 1+ \varepsilon$. After applying the rotation formula for this case one obtains $c = b(1+ \varepsilon) + [1 - (1 - \varepsilon)]a= - b\varepsilon^2 -a\varepsilon$. It is convenient to express this condition in the form

$$a + b\varepsilon + c\varepsilon^2 = 0.$$

In the second case, the triangle ACB has a counter-clockwise orientation and we have

$$a + c\varepsilon + b\varepsilon^2 = 0.$$

Therefore, the triangle is equilateral if and only if one of the above equalities is verified. This is equivalent to

$$(a + b\varepsilon + c\, \varepsilon^2)(a + c\varepsilon + b\varepsilon^2) = 0.$$

After multiplication and using the algebra of cube roots of unity, one obtains the required result.

3.3. An IMO problem with equilateral triangles.[5] We are given a triangle $A_1A_2A_3$ and a point P_0 in its plane. We define $A_m = A_{m-3}$ for all m ≥ 4. One constructs a sequence of points P_0, P_1, P_2, ... such that $P_{\kappa+1}$ is obtained from P_κ by a clockwise rotation about $A_{\kappa+1}$ by an angle of 120^0, for all k = 0, 1, 2, ... Prove that if $P_{1986} = P_0$ then the triangle $A_1A_2A_3$ is equilateral.

Solution. Let us assume that $\Delta A_1A_2A_3$ has a counter-clockwise orientation and let us denote by z_0, z_1, z_2, ... the affixes of the points P_0, P_1, P_2, ... respectively. Then, a clockwise rotation of 120^0 requires to use the complex number $\omega = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \varepsilon^2$.

By the above considerations we have:

$z_1 = z_0\,\varepsilon^2 + (1 - \varepsilon^2)a_1$

$z_2 = z_1\varepsilon^2 + (1 - \varepsilon^2)a_2 = z_0\varepsilon + (\varepsilon^2 - \varepsilon)a_1 + (1 - \varepsilon^2)\,a_2$

$z_3 = z_2\varepsilon^2 + (1 - \varepsilon^2)a_3 = z_0 + (\varepsilon - 1)a_1 + (\varepsilon^2 - \varepsilon)a_2 + (1 - \varepsilon^2)\,a_3$

$\quad = z_0 + (\varepsilon - 1)(a_1 + \varepsilon a_2 + \varepsilon^2 a_3).$

An easy induction upon n shows that after n cycles of three rotations one obtains that P_{3n} is represented by

$$z_{3n} = z_0 + n(1 - \varepsilon)(a_1 + \varepsilon a_2 + \varepsilon^2 a_3).$$

Thus, $z_{1986} = z_{662.3} = z_0 + 662(1 - \varepsilon)(a_1 + \varepsilon a_2 + \varepsilon^2 a_3)$.

Therefore, if $z_{1986} = z_0$, one has $a_1 + a_2\varepsilon + a\_3\varepsilon^2 = 0$. The required result follows.

3.4. An IMO problem which involves rotations.[6] Let ABC be a triangle. The triangles ABR, BCP and CAQ are drawn externally on the sides AB, BC, CA respectively, such that $\angle PBC = \angle CAQ = 45^0$, $\angle BCP\} = \angle QCA = 30^0$ and $\angle RBA = \angle RAB = 15^0$. Show that QR = RP and $\angle QRP = 90^0$.

This problem is more difficult and the students did not attempt at the time a solution using complex numbers. Nevertheless, this method makes it more accessible.

Solution. Too many angles in this problem! The advantage is that all are a multiple of 15^0. We shall use complex numbers in a very special way. Take the origin O of the complex plane as the point R, so R ≡ O, and the real axis to be parallel to the side BA. Also, we may assume that OA = OB = 1. (See Fig 1)

[5] This was Problem 2 in the 27th IMO, 1986.

[6] Problem 3 in the 17th IMO, 1975.

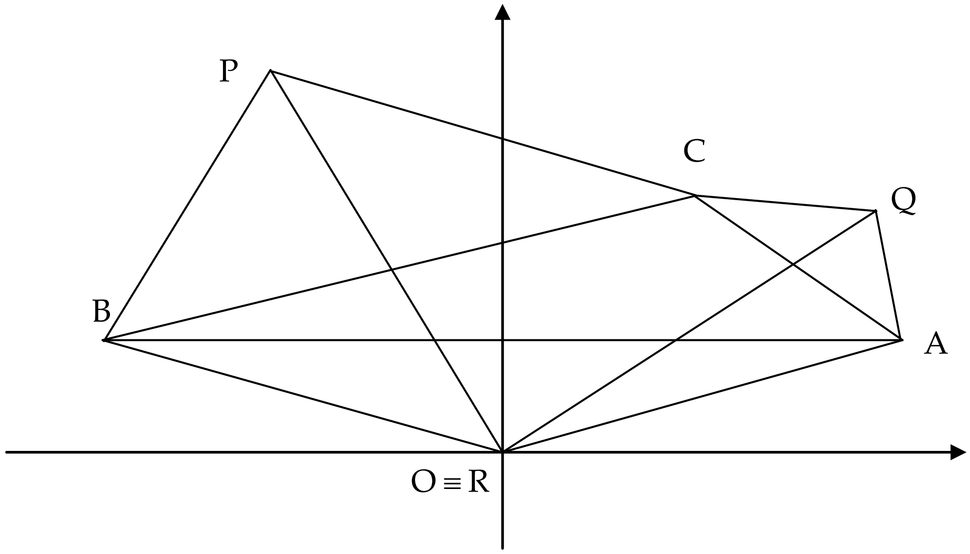


Figure 1

Since all angles considered are multiples of 15^0, let denote $\omega = \cos 15^0 + i\sin 15^0 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$. In this way, $a = \omega$ and $b = \omega^{11}$. Choosing the orientation of ΔABC as in (Fig 1) it is easy to see that the point C is obtained from Q by some rotation about A. So the complex number q is uniquely determined by the conditions:

(1) $$c - a = \frac{AC}{AQ}\,\omega^3(q - a),$$

(2) $$q - c = \frac{QC}{AC}\,\omega^2(a - c).$$

In the same way, the number p is completely determined by the conditions:

(3) $$b - c = \frac{BC}{PC}\,\omega^2(p - c),$$

(4) $$p - b = \frac{BP}{BC}\,\omega^3(c - b),$$

By the similarity of the triangles $\Delta AQC \sim \Delta BPC$ and then using the law of sines one obtains the proportionalities

$$\frac{AQ}{QC} = \frac{BP}{PC} = \frac{\sin 30^0}{\sin 45^0} = \frac{1}{\sqrt{2}}.$$

After multiplication of equalities (1) and (2) one has

$$q - c = \sqrt{2}\,\omega^5(a - q) \Leftrightarrow q(1 + \sqrt{2}\,\omega^5) = c + \sqrt{2}\,\omega^5 a.$$

Taking into account that $a = \omega$, finally we have

$$q = \frac{c + \sqrt{2}\omega^6}{1 + \sqrt{2}\omega^5}.$$

Similar computations with (3), (4) and using $b = \omega^{11}$, give the equality

$$p = \frac{(c + \sqrt{2}\omega^6)\omega^5}{\sqrt{2} + \omega^5}.$$

It is obvious that $1 + \sqrt{2}\omega^5 \neq 0$, $\sqrt{2} + \omega^5 \neq 0$ and $c + \sqrt{2}\omega^6 \neq 0$.

Now we have to prove that $p = iq$. Since $i = \omega^6$, this is successively equivalent to

$$\frac{(c + \sqrt{2}\omega^6)\omega^6}{1 + \sqrt{2}\omega^5} = \frac{(c + \sqrt{2}\omega^6)\omega^5}{\sqrt{2} + \omega^5} \Leftrightarrow 1 + \sqrt{2}\omega^5 = \omega(\sqrt{2} + \omega^5) \Leftrightarrow 1 - \omega^6 = \sqrt{2}\,(1 - \omega^4)$$

Again, since $\omega^6 = i$, we have $1 - \omega^6 = 1 - i = \sqrt{2}\,\omega^{21}$. Hence, the required equality is equivalent to

(5) $$\omega^{20} + \omega^4 = 1.$$

We can prove this in two ways. An easy computation shows that $\omega^{20} = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$ $= \frac{1}{2} - i\frac{\sqrt{3}}{2}$ and $\omega^4 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The conclusion follows.

The equality (5) can also be deduced from the picture bellow:

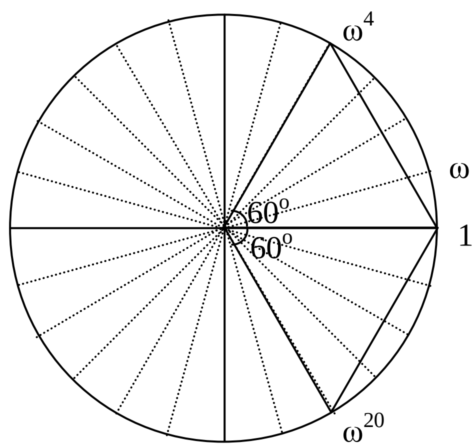


Figure 2

3.5 Cyclic points. It is known that any three points that are not collinear, lie on a circle. We say that the points A, B, C, D are cyclic if they lie on a circle. This condition can be expressed in terms of complex coordinates a, b, c, d of the points. The points A, B, C, D, which are not collinear, are cyclic if and only if their complex coordinates satisfy the condition

$$\frac{a-c}{b-c} : \frac{a-d}{b-d} \in \mathbf{R}.$$

This follows immediately from the considerations of 3.1. Given the points A, B, C, D, The point D lies on the circumcircle of the triangle ABC if and only if $\angle ACB = \angle ADB$. This means that the complex numbers $\frac{a-c}{b-c}$ and $\frac{a-d}{b-d}$ either have the same arguments or their arguments differ by π. Since they can have different moduli it follows that

$$\frac{a-c}{b-c} = \lambda \frac{a-d}{b-d},$$

where λ is a real number, positive or negative. So, the stated result follows.

3.6 A problem with cyclic points. Let ABC be a triangle, D be the foot of the altitude from A and K an arbitrary point on the segment AD. The perpendicular projections of the point D on the lines BA, BK, CA, CK are the points M, N, P, Q respectively. Show that the quadrilateral MNPQ is cyclic.

Solution. In this problem we have four perpendiculars from D. Hence it is convenient to choose coordinates such that D is the origin, the line BC the OX axis and DA the OY axis (See Figure 1) Therefore, the given points A, B, C, K have complex coordinates ai, b, c, k, respectively, where a, b, c $\in$ **R**.

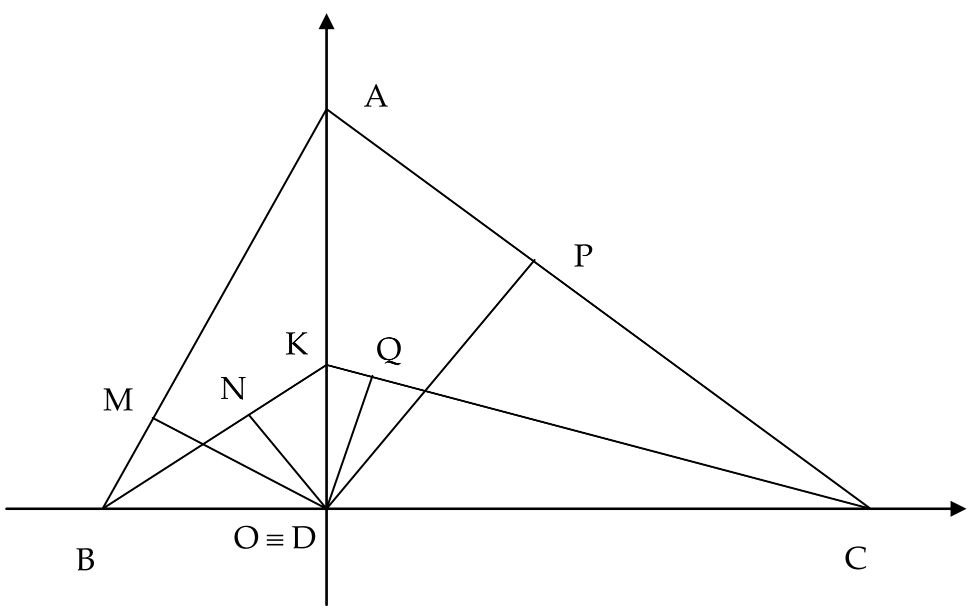


Figure 1

To obtain the coordinate m of the point M we will use the considerations from 2.1. and 3.1. The equation of the line AB is

$$m(a + bi) - \overline{m}(b - ai) = 2abi.$$

The condition of orthogonality between AB and DM is

$$m(a + bi) - \overline{m}(b - ai) = 0.$$

Solving the system given by these equations one obtains

$$m = \frac{abi}{b+ai} = \frac{ab}{a-bi}.$$

By changing in the above considerations the point A by k one obtains

$$n = \frac{bk}{k-bi}.$$

In the same way one obtains the coordinates p and q:

$$p = \frac{ck}{k-ci} \text{ and } q = \frac{ac}{a-ci}.$$

The condition of cyclic points as given in 3.5 is

$$\frac{m-p}{n-p} : \frac{m-q}{n-q} \in \mathbf{R}.$$

We replace m, n, p, q in the above and one has

$$\frac{m-p}{n-p} : \frac{m-q}{n-q} = \frac{\left(\frac{ab}{a-bi} - \frac{ck}{k-ci}\right) \cdot \left(\frac{bk}{k-bi} - \frac{ac}{a-ci}\right)}{\left(\frac{bk}{k-bi} - \frac{ck}{k-ci}\right) \cdot \left(\frac{ab}{a-bi} - \frac{ac}{a-ci}\right)}$$

$$= \frac{\left[ak(b-c) - bc(a-k)i\right]\left[ak(b-c) + bc(a-k)i\right]}{a^2k^2(b-c)^2}$$

$$= \frac{\left[ak(b-c)\right]^2 + \left[bc(a-k)\right]^2}{a^2k^2(b-c)^2} \in \mathbf{R}$$

This proves the required result.

4. The geometry of a triangle

4.1. Remarkable points in a triangle. Let ABC be a triangle and a, b, c be the affixes of its vertices, respectively. Let G, H, O, I be its centroid, orthocentre, circumcentre and the incentre, respectively. Our first aim is to find the complex coordinates of these points. Then we will proceed using them in problem solving.

The **centroid** G has the complex coordinate

$$g = \tfrac{1}{3}(a + b + c).$$

Indeed, this result has a vectorial character. It is known that G lies on the median AA' such that $\overrightarrow{AG} = 2\,\overrightarrow{GA'}$. Since $a' = \tfrac{1}{2}(b + c)$ it follows by the formula (3) that

$$g = \frac{1}{3}a + \frac{2}{3}a' = \frac{1}{3}a + \frac{2}{3}\,\frac{b+c}{2} = \frac{1}{3}(a + b + c).$$

If the circumcentre O is precisely the origin of the complex plane, then the **orthocentre** H has complex coordinate h = a + b + c. To prove this formula, it is enough to prove the following equality of vectors,

$$\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC},$$

which is known as Sylvester's equality. We give a nice proof for it. The idea is to compute the sum of vectors $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$ and then to see where is its end point H. The sum can be computed in the form $\overrightarrow{OA} + (\overrightarrow{OB} + \overrightarrow{OC})$ and we denote this sum by $\overrightarrow{OX}$, without stating yet where the point X is. The vector $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{OC}$ is such that $OD \perp BC$ because OBCD is a rhombus (see Figure 1).

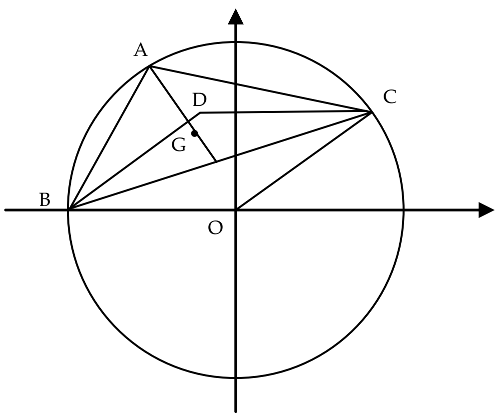


Figure 1

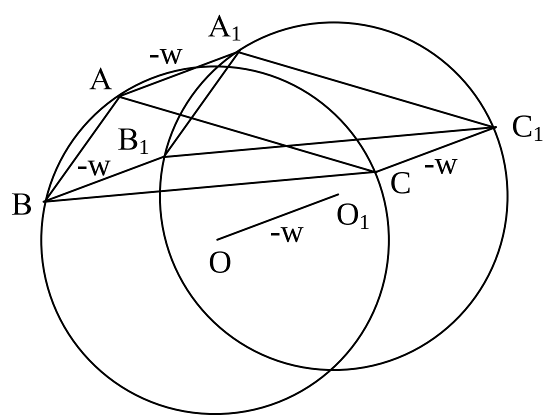


Figure 2

Then, the sum $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{OD}$ is such that OAXD is a parallelogram and we obtain that OD // AX. It follows that $AX \perp BC$ and so X lies on the altitude from vertex A in the triangle ABC. This procedure can be repeated by grouping the sum of vectors in other two different ways and we finally obtain that X lies on all three altitudes, so $X \equiv H$. Now, by using complex numbers instead of vectors, the required equality h = a + b + c follows.

If the circumcentre O of ΔABC is not at the origin and if it has complex coordinate w then the orthocentre H has complex coordinate h = a + b + c - 2w. To prove this, one may use a translation of vector -w to obtain a triangle with vertices a - w, b - w, c - w and the circumcentre at the origin of the coordinate axes. This triangle has its orthocentre in a point H' such that h' = (a - w) + (b - w) + (c - w) = a + b + c - 3w (See Fig. 2) By a new translation of vector w, which moves back to the original triangle, its orthocentre will be in H and h = h' + w = a + b + c - 2w.

The **incentre** I of a triangle can also be computed in complex coordinates by using the complex coordinates of the vertices and the lengths of the sides. Then the complex coordinate κ of the incentre I is given by the formula

$$\kappa = -\frac{AB \cdot a + BC \cdot b + CA \cdot c}{AB + BC + CA}$$

Proof. Let AD be the angle bisector of the $\angle BAC$, where the point D lies on the side BC. Then, by the bisector theorem the following equality BD : CD = AB : AC holds. Using formula (3) from Section 2.1. it follows that

$$d = \frac{BC \cdot b + CA \cdot c}{BC + CA}$$

The length of the segment BD is BD = $\frac{BC \cdot AB}{AC + AB}$. We apply again the bisector theorem for the bisector AI in triangle ABD and using the same formula (3) obtain

$$\kappa = \frac{1}{1 + \frac{CA + AB}{AB}} \cdot a + \frac{\frac{CA + AB}{AB}}{1 + \frac{CA + AB}{AB}} \cdot \frac{1}{CA + AB} \cdot (CA \cdot b + AB \cdot c) = \frac{AB \cdot a + BC \cdot b + CA \cdot c}{AB + BC + CA}$$

4.2. The nine points circle. In a given triangle ABC we denote by A', B', C' the midpoints of the sides BC, CA AB, by A'', B'', C'' the feet of the altitudes from A, B, C and by H_A, H_B, H_C the midpoints of the segments AH, BH and CH respectively. The nine points

$$A', B', C', A'', B'', C'', H_A, H_B, H_C$$

are on a circle (the *nine point circle* or *Euler circle* of the triangle ABC) and the centre of this circle is the midpoint of the segment OH.

Proof. We may assume that the circumcentre of ΔABC is the origin O of the complex plane and its circumradius is R, so that |a| = |b| = |c| = R. The complex coordinate of H is h = a + b + c and the coordinates of the points A', B', C', A'', B'', C'', H_A, H_B, H_C are a' = (b + c)/2, b' = (c + a)/2, c' = (a + b)/2, h_A = a + (b + c)/2, h_B = b + (c + a)/2 and h_C = c + (a + b)/2. Let Ω denote the midpoint of the segment OH. Its coordinate is ω = (a + b + c)/2. By computing distances we obtain

$$|a' - \omega| = \left|\frac{b + c}{2} - \frac{a + b + c}{2}\right| = \frac{|-a|}{2} = \frac{R}{2},$$

$$|h_A - \omega| = \left|a + \frac{b + c}{2} - \frac{a + b + c}{2}\right| = \frac{|a|}{2} = \frac{R}{2},$$

and the analogous for the vertices B and C. These prove that the six points A', B', C', A'', B'', C'', H_A, H_B, H_C are on the circle with centre Ω and of radius $\frac{R}{2}$.

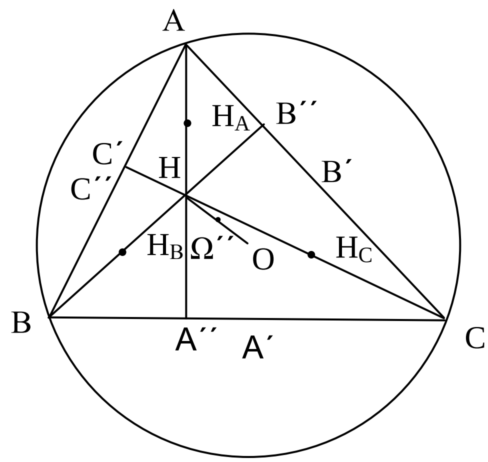


Figure 1

As for the remaining three points, we will use a geometric arguments coming from (Figure 1). Since Ω is the midpoint of OH it is at equal distance from A' and A''. So ΩA'' = 1/2. A similar argument is works for the points B'' and C''.

4.3. Two problems about triangles.

Problem 1. We are given an acute angled triangle ABC and let H_A, H_B, H_C be the feet of the altitudes from A, B, C respectively. The angle bisectors of $\angle BH_CC$ and $\angle BH_BC$ meet at a point K, the angle bisectors $\angle CH_CA$ and $\angle AH_AC$ meet at a point L and the angle bisectors $\angle BH_BA$, $\angle AH_AB$ meet at a point M. Show that if the triangles ΔABC and ΔKLM have the same orthocentre then AB = BC = CA.

Solution. Let consider the circle with AC as diameter. It passes through the points H_A and H_C. In this circle, the point L is the midpoint of that semicircle which does not contain H_A, H_C. Let B' be the midpoint of AC. Then B'L= AC/2 and B'L $\perp$ AC. It follows that L is the centre of the square constructed externally on the side AC. In the same way K, M are the centres of the squares constructed externally on the sides BC, AB respectively (See Figure 1)

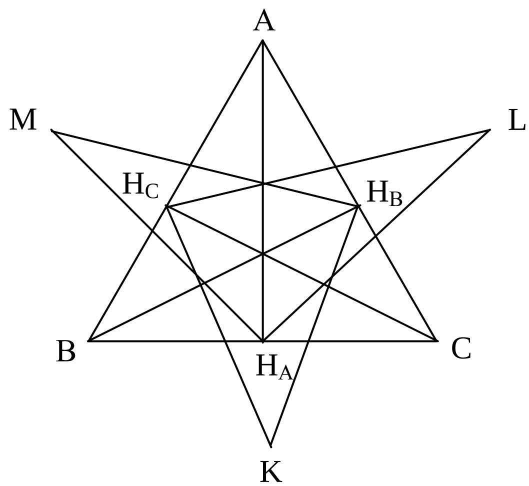


Figure 1

Now, considering the complex numbers A(a), B(b), C(c) one obtains for the points K, L, M the following coordinates:

$$k = \frac{b+c}{2} + i\,\frac{b-c}{2},$$

$$l = \frac{c+a}{2} + i\,\frac{c-a}{2},$$

$$m = \frac{a+b}{2}\ i\ \frac{a-b}{2}.$$

From these equalities it follows that k + l + m = a + b + c.

The orthocentre H of ABC is given by h = a + b + c. Let w be the circumcentre of the triangle KLM. Then, its orthocentre is given by k + l + m - 2w = a + b + c - 2w = h - 2w. It is given that this point is H. It follows that w = 0. So, the triangles ABC and KLM have the same circumcentre. From here one gets OK = OL = OM. Standard trigonometric computations give

$$OK = R\cos A + a/2;\ OL = R\cos B + b/2;\ OM = R\cos C + c/2,$$

where R is the circumradius of ΔABC. Using the sine theorem and the above relations one obtains

$$\sin A + \cos A = \sin B + \cos B = \sin C + \cos C,$$

which can be transformed into

$$\sin(A + \frac{\pi}{4}) = \sin(B + \frac{\pi}{4}) = \sin(C + \frac{\pi}{4}).$$

From the last equalities one has either $A + B = C = \frac{\pi}{2}$ or A = B = C. Since the triangle is acute, only the second case is possible.

Problem 2. Let ABC be a triangle and H be its orthocentre. On the circumcircles of the triangles BCH, CHA, AHB are taken the points A', B', C', respectively such that HA' = HB' = HC'. Let K, L, M be the othocentre of the triangle BA'C, CB'A, AC'B, respectively. Show that the triangles A'B'C' and KLM have the same orthocentre.

Solution. It is known that the reflection of H in a side of the triangle lies on the circumcircle. This can be proved as follows: if the altitude AH intersects the side BC in X and the circumcircle in Y then the triangles HBX and YBX are equal. It follows that HX = YX (See Fig 1.) and this proves the property. Since the circumcircle of ΔABC passes through B, C and Y it follows that the circumcircle of ΔBCH is the reflection of the first circle in the line BC. The reflection of the circumcentre O is the point W defined by $\overrightarrow{OW} = \overrightarrow{OB} + \overrightarrow{OC}$. It follows that w = b + c. Moreover, |w - b| = |w - c| = |w - a'| = R, where R is the circumradius of ΔABC.

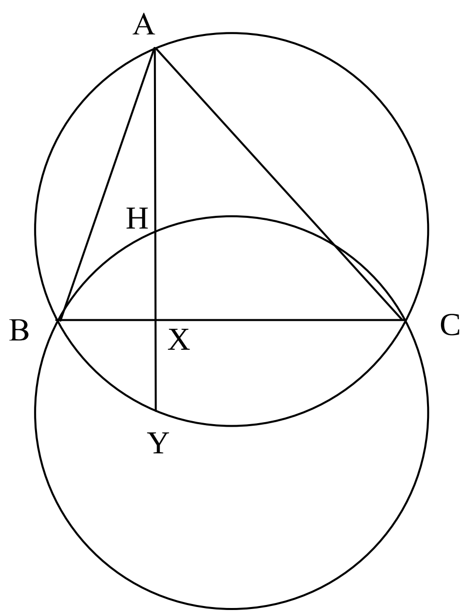


Figure 1

By the formula from 4.1 we obtain k = a' + b + c - 2w = a' - (b + c). In the same way, l = b' - (c + a) and m = c' - (a + b). Summing up these equalities one obtains

$$k + l + m = (a' + b' + c') - 2(a + b + c) = (a' + b' + c') - 2h.$$

The circumcentre of ΔA'B'C' is H. Therefore, again by 4.1, $h_{A'B'C'} = (a' + b' + c') - 2h = k + l + m$. Since |k| = |a' - w| = R it follows that the circumcentre of ΔKLM is O. Hence, $h_{KLM} = k + l + m$ and the required result follows.

5. Proposed problems

Problem 5.1. Show that there does not exist an equilateral triangle whose vertices are the corners of squares of an infinite chessboard.

Solution. An infinite chessboard can be regarded as a plane surface endowed with an orthogonal system of coordinates. Then, the corners of the squares can be regarded to be the points of the plane whose coordinates are integer numbers. Passing to complex numbers, they are exactly those numbers which can be written under the form $z = a + bi$ where $a, b \in Z$.

We mention that the set $\mathbf{Z}[i] = \{a + bi \mid a, b \in \mathbf{Z}\}$ is of great importance in Geometry, Algebra and Number Theory. It is called the ring of Gauss integers.

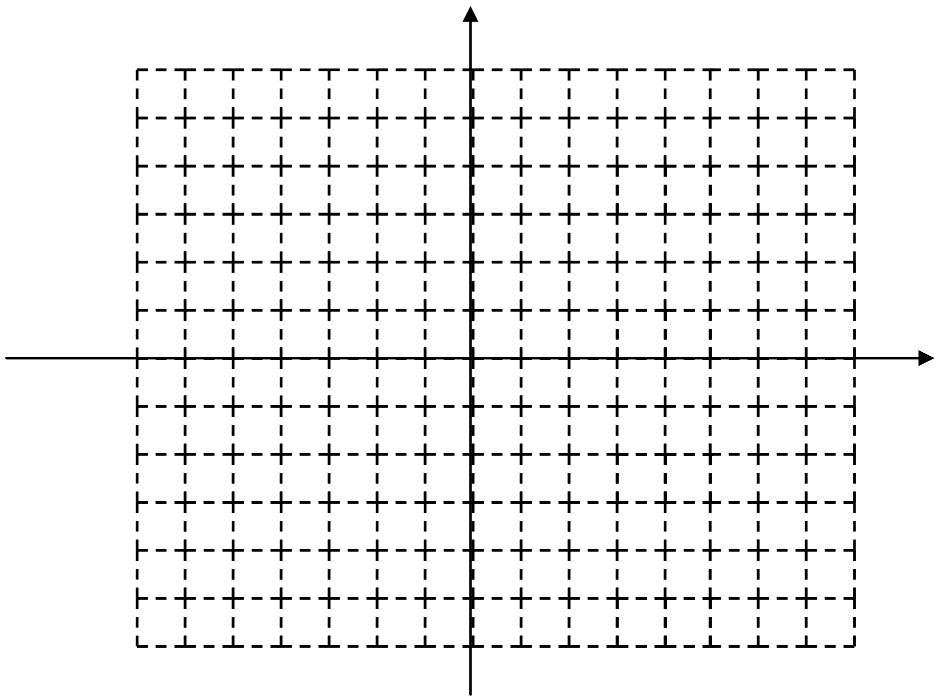


Figure 1

Now, let assume that there exists an equilateral triangle ABC whose vertices are Gauss integers. That is, there exist complex numbers $a, b, c \in \mathbf{Z}[i]$ such that $a + b\varepsilon + c\varepsilon^2 = 0$. Using the equality $\varepsilon^2 = -(\varepsilon + 1)$ it follows that $\varepsilon(b - c) = c - a$. Hence, one obtains for ε an expression of the form

$$\varepsilon = \frac{c-a}{b-c} = \alpha + \beta i,$$

where α, β are rational numbers. This contradicts the exact formula for ε which says that its imaginary part is $\frac{\sqrt{3}}{2}$, an irrational number.

Problem 5.2. Assume that the plane is the disjoint union of a family of equilateral triangles. Show that there does not exist a square whose vertices are the meeting points of the lines comprising the boundaries of these triangles.

Solution. Like in the previous problem, we consider the plane to be endowed with the structure of a complex plane. We need a description of its covering with disjoint equilateral triangles. For this we take the regular hexagon inscribed in the unit circle such that its vertices are the points 1, $1 + \varepsilon$, ε, -1, $\varepsilon^2 = -(1+ \varepsilon)$, $1 + \varepsilon^2 = - \varepsilon$, and join the vertices with the centre of the circle (See Figure 1). One obtains six equilateral triangles. Then we move the hexagon by a translation of vector $1 + (1 + \varepsilon) = 2 + \varepsilon$ to obtain a new hexagon and six more equilateral triangles. The procedure can be continued indefinitely by using other translation, as given by the sums of two consecutive vertices. So we obtain a covering of the plane with disjoint equilateral triangles (See Figure 2). It is easy to see that the vertices of these triangles are exactly the complex numbers from the set $Z[\varepsilon] = \{a + b\varepsilon \mid a, b \in \mathbf{Z}\}$.

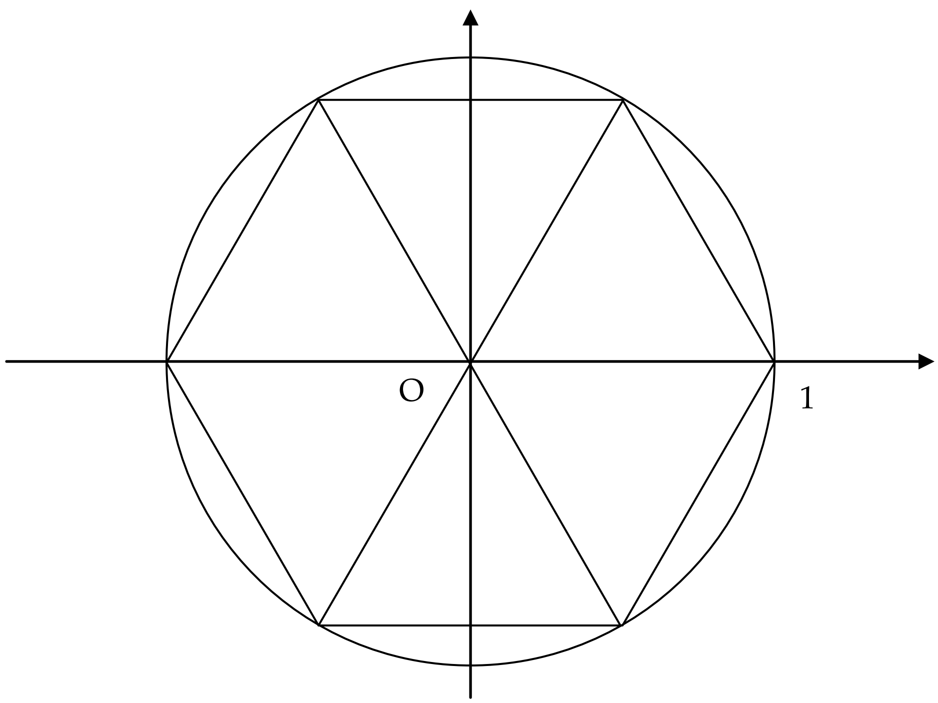


Figure 1

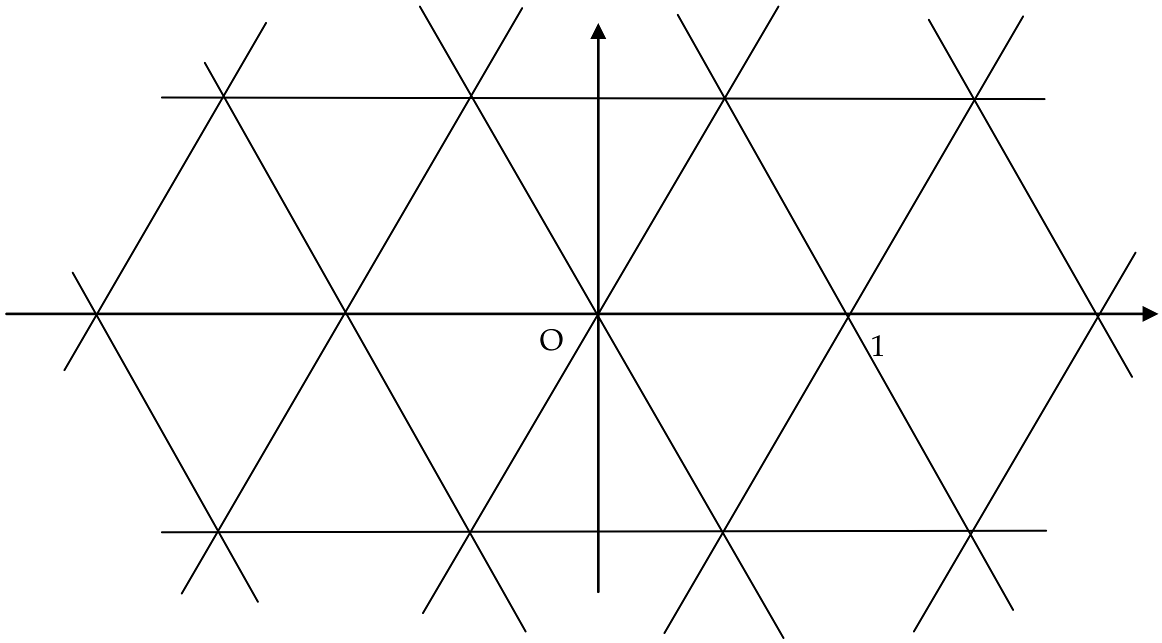


Figure 2

Using this description of the vertices one can consider the problem. Assume by contradiction that there exists a square whose vertices are in the set **Z**[ε]. Then there are complex numbers z, u, v $\in$ **Z**[ε] such that $i = \frac{u - z}{v - z}$. The numbers u - z and v - z are also in the set **Z**[ε]. So, one obtains an equality of the form

$$m + n\varepsilon = i(p + q\varepsilon),$$

where m, n, p, q are integers. Using the exact value of ε one obtains

$$m - \frac{n}{2} + I\,\frac{n\sqrt{3}}{2} = -\frac{q\sqrt{3}}{2} + i(p - \frac{q}{2}),$$

which is in contradiction to the condition m, n, p, q $\in$ **Z**.

Problem 5.3.[7] We are given a convex pentagon which satisfies the conditions:

(a) all interior angles are congruent,

(b) the lengths of all sides are rational numbers.

Show that the pentagon is regular.

Solution. Let $A_1A_2A_3A_4A_5$ be the given pentagon and assume that it has the counter-clockwise orientation (see Fig.1). Each side $\overrightarrow{A_1A_2} = \overrightarrow{a_1}$, $\overrightarrow{A_2A_3} = \overrightarrow{a_2}$, ... , $\overrightarrow{A_5A_1} = \overrightarrow{a_5}$ can be considered as a vector. In the plane, these vectors can be translated to have a common origin O and such that the vector a_1 is along the real axes (see Fig.2).

[7] This is Problem 2 from 18th Balkan Mathematical Olympiad, 2001.

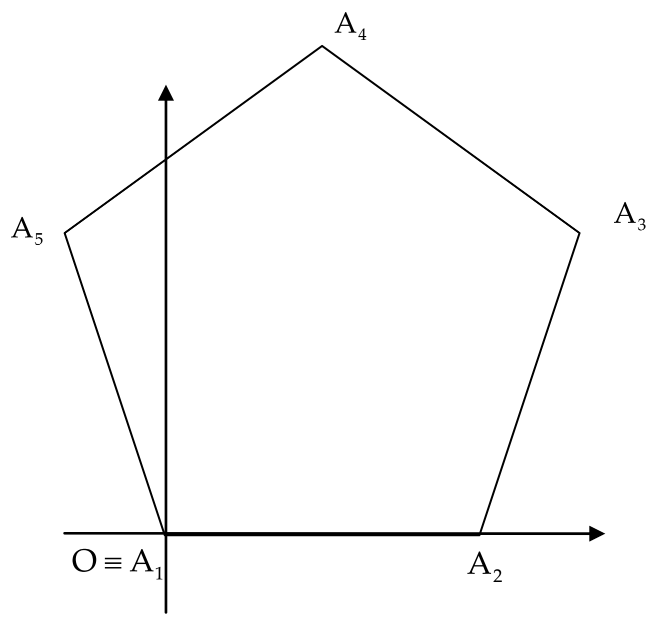


Figure 1

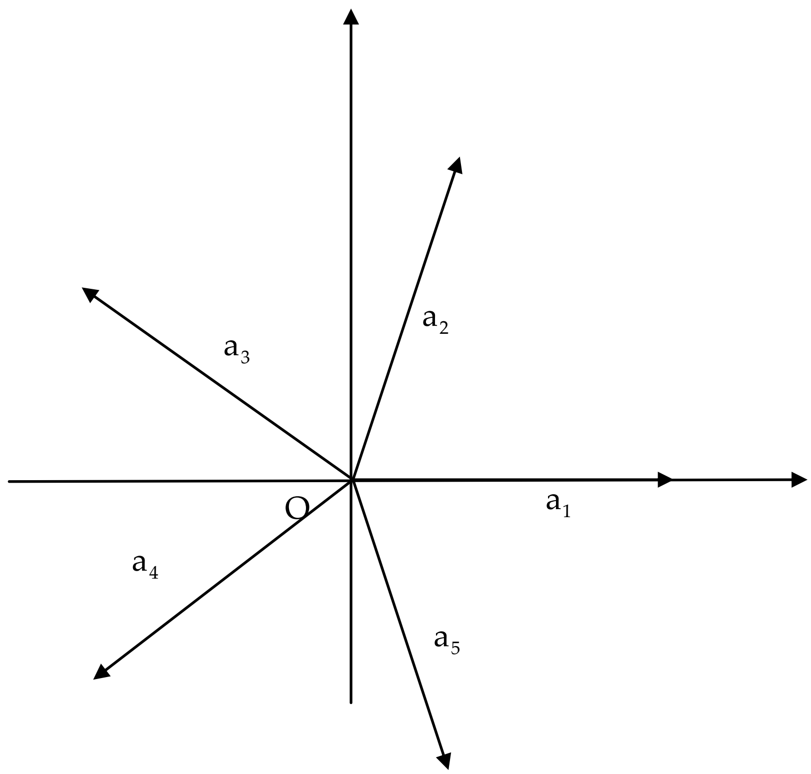


Figure 2

The angle between two consecutive vectors is $72^0 = \frac{2\pi}{5}$ and $\sum_{k=1}^{5} \overrightarrow{a_k} = 0$. It is natural to introduce the fifth complex root of unity, $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Using complex numbers instead of vectors we obtain the equalities

$$a_1 = |a_1|,\ a_2 = |a_2|\omega,\ a_3 = |a_3|\omega^2,\ a_4 = |a_4|\omega^3,\ a_5 = |a_5|\omega^4.$$

So, we obtain the equality

$$|a_1| + |a_2|\omega + |a_3|\omega^2 + |a_4|\omega^3 + |a_5|\omega^4 = 0.$$

It is an equation with rational coefficients $|a_1|, \dots, |a_5|$, which is satisfied by the complex number ω. But ω is a root of the irreducible integer polynomial

$$\Phi_5(X) = X^4 + X^3 + X^2 + X + 1.$$

Hence $|a_1| = |a_2| = \dots = |a_5|$, and the result follows.

Problem 5.4.[8] We are given a convex octagon which satisfies the conditions:

(a) all interior angles are congruent

(b) the lengths of all its sides are rational numbers.

Show that the octagon has a centre of symmetry.

Solution. Like in the previous problem, we assume that the sides of the octagon are vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, \dots, \overrightarrow{a_8}$ which have rational lengths, say $|a_1|, |a_2|, \dots, |a_8|$. Put all these vectors in the same origin O of the complex plane and associate with them the complex numbers $a_1, a_2, \dots, a_8$.

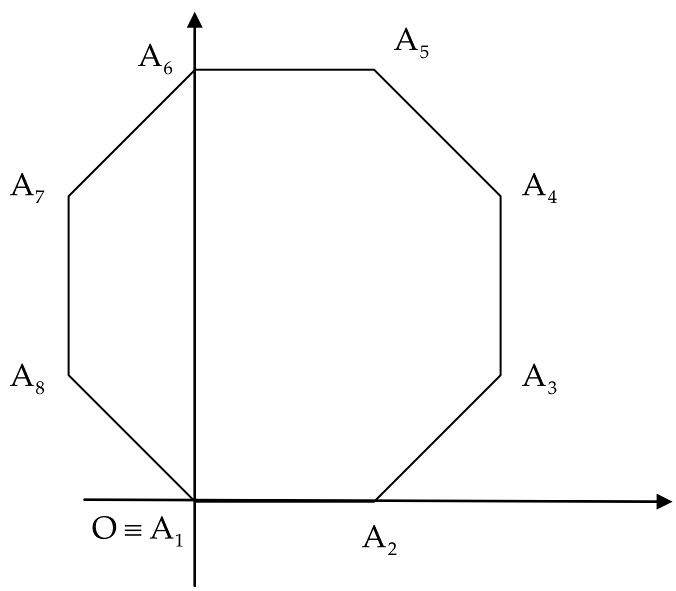


Figure 1

[8] A problem from the Russian Mathematical Olympiad.

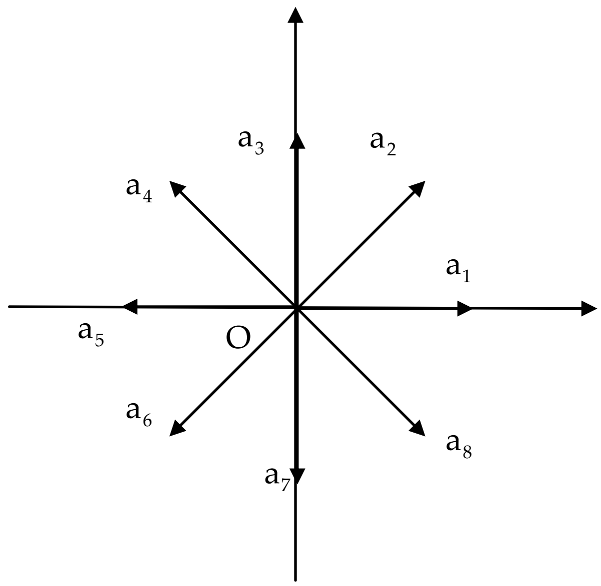


Figure 2

The angle between two consecutive vectors is $\varphi = 45^0 = \frac{\pi}{4}$. Let us denote $\omega = \cos\frac{\pi}{4} +$ is in $\frac{\pi}{4}$. Then, the following equality holds:

$$a_1 + a_2\omega + a_3\omega^2 + ... + a_8\omega^7 = 0.$$

Since $\omega^4 = -1$, the equality becomes

$$(a_1 - a_5) + (a_2 - a_6)\omega + (a_3 - a_7)\omega^2 + (a^4 - a^8)\omega^3 = 0.$$

Hence, we deduce that ω is a root of a rational polynomial of degree at most 3. It is known that the minimal polynomial of ω over **Q** is $\Phi_8(X) = X^4 + 1$. Therefore $a_1 = a_5$, $a_2 = a_6$, $a_3 = a_7$ and $a_4 = a_8$. These equalities show that the pairs of opposite sides of the octagon define four parallelograms. These parallelograms have a common centre of symmetry which turns to be a centre of symmetry of the octagon.

Problem 5.5. The equilateral triangles $A_2B_1A_3$, $A_3B_2A_1$ and $A_1B_3A_2$ are drawn externally on the sides of a triangle $A_1A_2A_3$.

a) Show that the lines A_1B_1, A_2B_2, A_3B_3 meet at a common point F.

b) Show that $\angle A_1FA_2 = \angle A_2FA_3 = \angle A_3FA_1 = 120^0$ (F is the *Fermat-Toricelli* point of the triangle $A_1A_2A_3$)

c) Show that $FB_1 = FA_2 + FA_3$, $FB_2 = FA_3 + FA_3$ and $FB_3 = FA_1 + FA_2$.

Solution. We assume that the triangle $A_1A_2A_3$ has the counter-clockwise orientation. Then the triangles $A_2B_1A_3$, $A_3B_2A_1$ and $A_1B_3A_2$ also have the counter-clockwise orientation (see Figure 1)

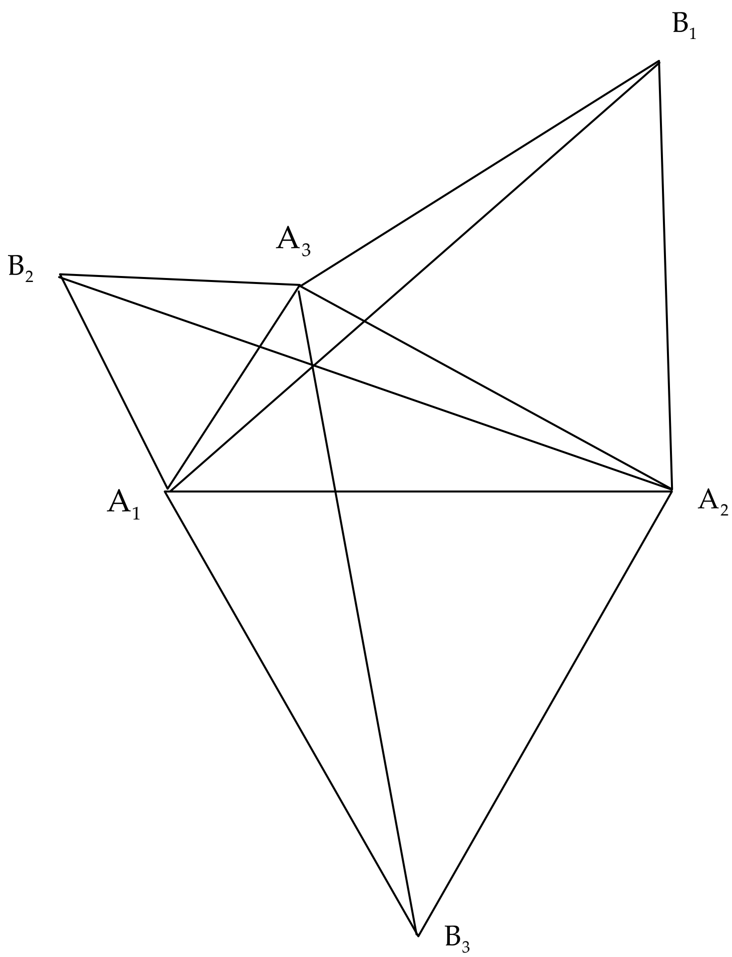


Figure 1

By the considerations from 3.2 the complex coordinates of the points B_1, B_2, B_3 are given by the formulas:

$$b_1 + a_3\varepsilon + a_2\varepsilon^2 = 0,$$

$$b_2 + a_1\varepsilon + a_3\varepsilon^2 = 0,$$

$$b_3 + a_2\varepsilon + a_1\varepsilon^2 = 0.$$

Summing up these three equalities and taking in account that $\varepsilon + \varepsilon^2 = -1$ one obtains the equality

$$b_1 + b_2 + b_3 = a_1 + a_2 + a_3.$$

The equations of the lines A_1B_1, A_2B_2, A_3B_3 can de written by using the considerations from 2.1 and one obtains the following linear equations in z and $\bar{z}$:

$$z(\bar{a}_1 - \bar{b}_1) - \bar{z}(a_1 - b_1) + (a_1\bar{b}_1 - \bar{a}_1 b_1) = 0,$$

$$z(\bar{a}_2 - \bar{b}_2) - \bar{z}(a_2 - b_2) + (a_2\bar{b}_2 - \bar{a}_2 b_2) = 0,$$

$$z(\overline{a}_3-\overline{b}_3)-\overline{z}(a_3-b_3)+(a_3\overline{b}_3-\overline{a}_3b_3)=0.$$

These three lines are concurrent if and only if there exist a point of the plane whose coordinate gives a solution (z, $\overline{z}$) of the above system. This condition means that the following determinant vanishes:

$$\begin{vmatrix} \overline{a}_1-\overline{b}_1 & a_1-b_1 & a_1\overline{b}_1-\overline{a}_1b_1 \\ \overline{a}_2-\overline{b}_2 & a_2-b_2 & a_2\overline{b}_2-\overline{a}_2b_2 \\ \overline{a}_3-\overline{b}_3 & a_3-b_3 & a_3\overline{b}_3-\overline{a}_3b_3 \end{vmatrix}=0$$

Summing up to the first the next two lines of the determinant and taking in account the fact that

$b_1 + b_2 + b_3 = a_1 + a_2 + a_3$, one obtains that the determinant vanishes if and only if the following equality holds:

$$a_1\overline{b}_1 + a_2\overline{b}_2+a_3\overline{b}_3 = \overline{a}_1b_1+\overline{a}_2b_2+\overline{a}_3b_3.$$

For the next point of the problem, we will prove more: the line segments A_1B_1 and A_2B_2 have equal lengths and the angle between them is 120^0. For this, it is sufficient to prove the equality $(b_1 - a_1)\varepsilon = b_2 - a_2$. Using the formula which gives the expressions of b_1, b_2 the equality follows immediately.

The last part of the problem is a consequence of the Ptolemy's theorem applied to the cyclic quadrilaterals $FA_2B_1A_3$, $FA_3B_2A_1$, $FA_1B_3A_2$. So, one obtains that for any point of the circumcircle of an equilateral triangle, the distance any one of the vertices equals the sum of distances to the other two vertices. This is a classical result, known as the theorem of Schooten.

Problem 5.6. Let ABCD be a cyclic quadrilateral and let H_A, H_B, H_C, H_D be the orthocentres of the triangles BCD, CDA, DAB and ABC, respectively. Show that the quadrilaterals ABCD and $H_AH_BH_CH_D$ are congruent.

Solution. Assume that the circumcentre O of the quadrilateral is the origin of the complex plane. Then the orthocentres of the triangles BCD, CDA, DAB, ABC have the complex coordinates $h_A = b + c + d$, $h_B = c + d + a$, $h_C = d + a + b$, $h_D = a + b + c$, respectively. Note that $h_A - h_C = a - b$, thus the vectors $\overrightarrow{AB}$ and $\overrightarrow{H_AH_B}$ are parallel, have the same length and different orientations. Using the same argument for the other pairs of consecutive sides of the quadrilateral, we obtain the desired conclusion.

Remark. Following the above solution one may obtain an interesting characterisation of the quadrilateral $H_AH_BH_CH_D$. Let S be the point corresponding to the complex number

$$s = \frac{a+b+c+d}{2}.$$

Then

$$a + h_A = b + h_B = c + h_C = d + h_D = 2s.$$

These equalities show that H_A, H_B, H_C, H_D are the reflections of the points A, B, C, D along the point S.

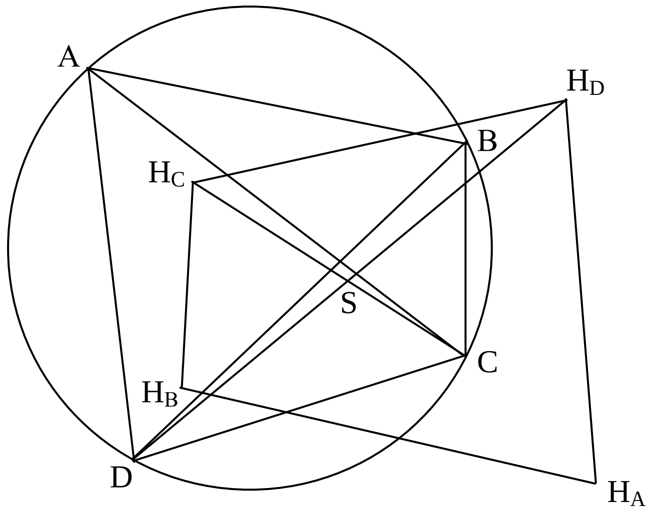


Figure 1

Problem 5.7. Let ABC be a triangle and P be an interior point such that $\angle PAC = \angle PBC$. Let K, L be the feet of the perpendicular projections from P on AC, BC, respectively and D be the midpoint of the segment AB. Show that DK = DL.

Solution. In this problem, the point C is irrelevant. We may assume that the midpoint of the segment AB is the origin of the complex plane, so that D = O and b = - a. Take an arbitrary point P in the plane and then construct the points K and L as in the problem: $\angle PAK = \angle PBL = \alpha$ and $PK \perp AK$, $PL \perp BL$ (see Figure 1)

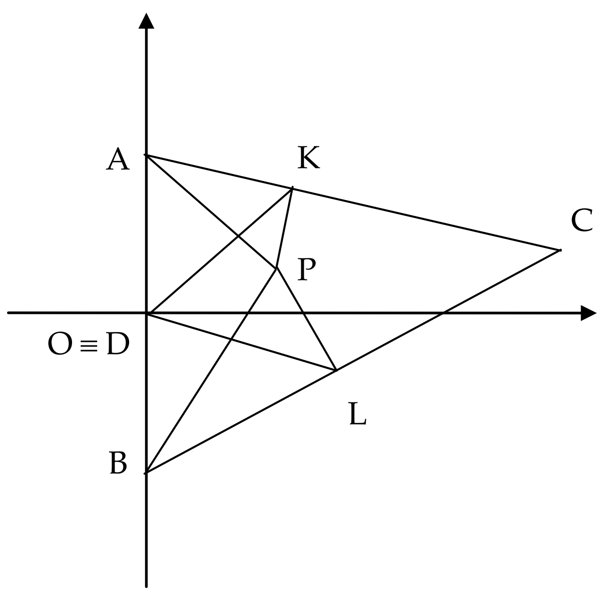


Figure 1

In terms of complex coordinates, the numbers k and l are defined by the following expressions:

$$p - k = i(a - k)\tan\alpha,$$

$$a + l = i(l - p)\cot\alpha.$$

From the above formulas one obtains for k and l the equalities

$$k = \frac{p - ai\tan\alpha}{1 - i\tan\alpha} \text{ and } l = \frac{p - ai\tan\alpha}{1 + i\tan\alpha}$$

It is now obvious that $k\bar{k} = l\bar{l}$, which means that $|k|^2 = |l|^2$.

Problem 5.8. Show that the line joining the midpoints of the diagonals of a quadrilateral circumscribed in a circle passes through the centre of this circle (Newton line of the quadrilateral).

Solution. Assume that the centre of the inscribed circle is the origin O of the complex plane and that its radius is 1. Let X, Y, Z, W be the tangency points of the sides AB, BC, CD, DA, respectively and x, y, z, w be their complex coordinates. Since $|x| = |y| = |z| = |w| = 1$ we have $\bar{x} = \frac{1}{x}$; $\bar{y} = \frac{1}{y}$; $\bar{z} = \frac{1}{z}$; $\bar{w} = \frac{1}{w}$. The idea of the solution is to compute the coordinates a, b, c, d in terms of x, y, z, w.

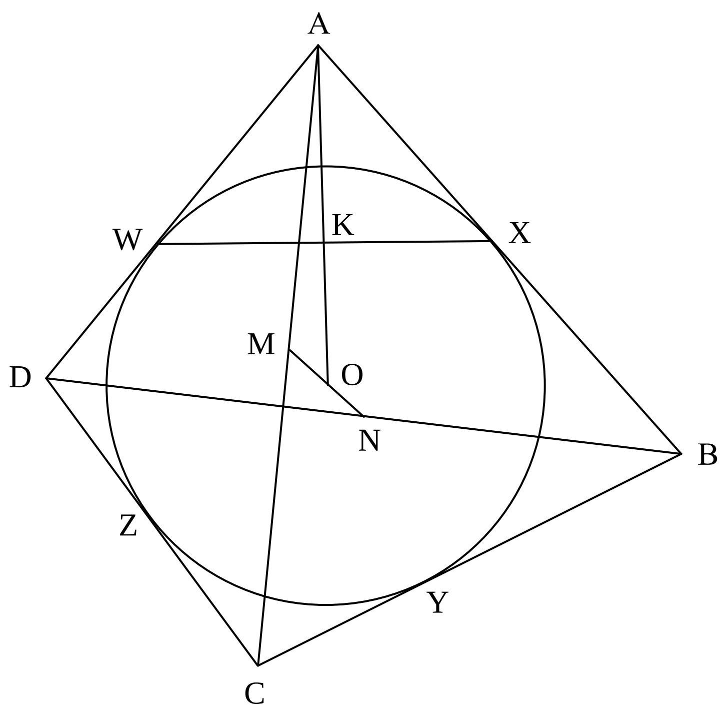


Figure 1

Let K be the midpoint of the segment XW. Since A is the meeting point of the tangents AX and AW the line AK is the perpendicular bisector of the segment XW and the points O, K, A are collinear (see the figure). The triangles ΔOKX and ΔOWA are similar and similarly oriented. By the condition 3.2 we have

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & k & x \\ 0 & w & a \end{vmatrix} = 0.$$

From this equation and taking into account that k = $\frac{x+w}{2}$ one obtains that the number a is given by

$$a = \frac{xw}{k} = \frac{2xw}{x+w}.$$

In the same way

$$b = \frac{2xy}{x+y};\ c = \frac{2yz}{y+z};\ d = \frac{2zw}{z+w}.$$

Let M be the midpoint of the diagonal segment AC. Then

$$m = \frac{xw}{x+w} + \frac{yz}{y+z} = \frac{1}{\bar{x}+\bar{w}} + \frac{1}{\bar{y}+\bar{z}}.$$

In the same way the midpoint N of BD has coordinate

$$n = \frac{xy}{x+y} + \frac{zw}{z+w} = \frac{1}{\bar{x}+\bar{y}} + \frac{1}{\bar{z}+\bar{w}}.$$

Accordingly to 2.1, the condition that the points M, O, N are collinear can be written in the form $m\bar{n} = \bar{m}n$. Using the expressions from above and having in mind that $x\bar{x}$ = ... = $w\bar{w}$ = 1, one has

$$m\bar{n} = \left(\frac{1}{\bar{x}+\bar{w}} + \frac{1}{\bar{y}+\bar{z}}\right)\left(\frac{1}{x+y} + \frac{1}{z+w}\right) = \frac{\bar{x}+\bar{y}+\bar{z}+\bar{w}}{(\bar{x}+\bar{w})(\bar{y}+\bar{z})} \cdot \frac{x+y+z+w}{(x+y)(z+w)}$$

$$= \frac{xyzw\,(x+y+z+w)(\bar{x}+\bar{y}+\bar{z}+\bar{w})}{(x+y)(y+z)(z+w)(w+x)}.$$

This expression is cyclic in x, y, z, w. Because n is obtained from m after a cyclic permutation, it follows that $\bar{m}n$ has the same form. The required result follows.

Problem 5.9.[9] In a given triangle ABC, let h_a be the length of the altitude from A, m_a is the length of the median from A and R, r are the circumradius and inradius, respectively. Show the inequality $\frac{R}{2r} \geq \frac{m_a}{h_a}$

and prove that the equality holds if and only if the triangle is equilateral.

Solution. Denote by S the area and by s the semiperimeter of the triangle. Then,

$$2rm_a \leq R_a \Leftrightarrow 2m_a \frac{S}{s} \leq \frac{2RS}{BC} \Leftrightarrow 2m_a \cdot BC \leq 2Rs.$$

[9] A problem from Romanian journal Gazeta Matematica,1981.

Assume that the circumcircle of ΔABC has the centre in the origin O of the complex plane and let a, b, c be the complex coordinates of the vertices. Then $|a| = |b| = |c| = R$. The left hand side of the required inequality can be computed as follows:

$$2m_a \cdot BC = 2|b - c||a - \frac{b+c}{2}| = |b - c||2a - b - c| = |(b - c)(2a - b - c)|$$

$$= |a(b - c) + b(a - b) + c(c - a)|$$

$$\leq |a||b - c| + |b||a - b| + |c||c - a| = 2Rs.$$

Equality occurs if and only if the numbers $a(b - c)$, $b(a - b)$, $c(c - a)$ have the same direction. This means that ΔABC is equilateral.

SEQUENCES

Péter Körtesi
University of Miskolc, Hungary

Section 1. Revision

This chapter is about sequences. It is recommended that the student reads first the notes on *Sequences – Level 1*, which mostly concern Arithmetic, Geometric and Harmonic Progressions. Here we shall first summarize some of the results from that chapter.

On *Sequences – Level 1* we worked with an intuitive description of a sequence. If we like an exact definition, here is how we do it:

Definition. An (infinite) *sequence* is a map $f : \mathbb{N} \to \mathbb{R}$ from the positive integers (natural numbers) to the real numbers.

It is customary not emphasize the function that defines a sequence and instead of writing f(1), f(2), f(3), ... for the images of the natural numbers, to simply denote sequence as, for example,

$$a_1, a_2, a_3, a_4, \ldots$$

This is denoted with the shorthand notation (a_n), or $(a_n)_{n\in\mathbb{N}}$ and a_n is called the *general term*.
Sometimes we talk of *finite sequences*. By this we mean an initial finite set of terms of an infinite sequence. For instance $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ is a finite sequence of 8 terms
A sequence may be described by giving the general term, e.g. $a_n = \frac{n^2}{n+1}$, or by a *recurrence relation* which specifies the way further terms of a sequence are obtained from previous ones. For example the Fibonacci sequence is described by

$$a_1 = 1,\ a_2 = 1 \text{ and } a_{n+2} = a_{n+1} + a_n \text{ (for } n \geq 1)$$

Examples of sequences given by linear recurrence relations are the arithmetic, geometric and harmonic progressions.

Recall that if in a sequence (finite or infinite) any three consecutive terms a_{n-1}, a_n and a_{n+1} satisfy the relation

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

then it is an arithmetic progression.
If in a sequence any three consecutive terms a_{n-1}, a_n and a_{n+1} satisfy the relation

$$a_n^2 = a_{n-1}a_{n+1}$$

then it is a geometric progression.
Finally, if in a sequence of non-zero terms any three consecutive terms a_{n-1}, a_n and a_{n+1} satisfy the relation

$$\frac{2}{a_n} = \frac{1}{a_{n-1}} + \frac{1}{a_{n+1}}$$

then it is a harmonic progression.

The same three progressions can be given recursively as follows:

$a_{n+1} = a_n + r$, where a_1, and r are given, is an arithmetic progression,

$a_{n+1} = a_n r$, where a_1 and r are given, is a geometric progression, and

$\frac{1}{a_{n+1}} = \frac{1}{a_n} + r$, where a_1 and r are given (and a is not a positive integral multiple of r), is a harmonic progression.

For the above three types of progressions, their general terms are given by $a_n = a + (n-1)r$, $a_n = ar^{n-1}$ and $a_n = \frac{a(a+r)}{a-(n-2)r} = \frac{a(a+r)}{a+(2-n)r}$ respectively.

Problem 1. Prove that an identity similar to the definition holds for three "equidistant terms" a_{n-k}, a_n, a_{n+k} for each of the three types of i.e.

$a_n = \frac{a_{n-k} + a_{n+k}}{2}$, $a_n^2 = a_{n-k}a_{n+k}$ and $\frac{2}{a_n} = \frac{1}{a_{n-k}} + \frac{1}{a_{n+k}}$, respectively.

The sum of the first n terms of an arithmetic progression is given by

$$S_n = a_1 + a_2 + a_3 + ... + a_n = \frac{n(a_1 + a_n)}{2} = \frac{n(2a + (n-1)r)}{2}.$$

Similarly for a geometric progression we have

$$S_n = a_1 + a_2 + a_3 + ... + a_n = \frac{a(1-r^n)}{1-r}$$

Prove the above formulas as simple exercise (hints are given in *Sequences* – *Level 1*)

Problem 2. Prove that for a geometric progression the product P_n of the first n terms is given by

$$P_n = a_1 a_2 a_3 ... a_n = a^n r^{\frac{n(n-1)}{2}}$$

Problem 3. Prove that for the sum of the reciprocals of first n terms of the harmonic sequence we will have:

$$R_n = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_n} = \frac{n}{2}\left(\frac{1}{a_1} + \frac{1}{a_n}\right) = \frac{n(2a + (3-n)r)}{2a(a+r)}$$

Typical elementary problems for arithmetic progressions

Problem 4. The first three terms of arithmetic progression are 20, 16.5 and 13. Find the fifteenth term.

Solution. The common difference is $r = 16.5 - 20 = -3.5$ and $a_1 = 20$. Thus $a_{15} = a + (15-1)r = 20 + 14 \cdot (-3.5) = -29$.

Problem 5. If the third term of an arithmetic sequence is 2, that is $a_3 = 2$, and the ninth term is 20, that is $a_9 = 20$, find the sixth term.

Solution. Plug the given information into the formula $a_n = a + (n-1)r$ and this gives: $a_3 = a + 2r = 2$ and $a_9 = a + 8r = 20$. This simultaneous system has solution $a = -4$ and $r = 3$ which means that $a_6 = a + 5r = -4 + 15 = 11$.

Observe that the problem could be solved more simply if using the formula for equidistant terms, as $a_6 = \dfrac{a_{6-3} + a_{6+3}}{2} = \dfrac{a_3 + a_9}{2} = \dfrac{2+20}{2} = 11$.

Typical elementary problem for geometric progressions

Problem 6. If the third term of a geometric sequence is 5 and the sixth term is −40, find the eight term

Solution. Using the formula for the n^{th} term we have $a_3 = ar^2 = 5$, $a_6 = ar^5 = -40$. Solving the system gives $r^3 = -8$, with the real solution $r = -2$ and from there $a = \dfrac{5}{4}$

Thus $a_8 = ar^7 = \dfrac{5}{4} \cdot (-2)^7 = -160$.

Perhaps the most famous problem on geometric progressions is the Chess Master problem:

Problem 7. A king promised to give to the chess master anything that he has if the master wins (which he easily did). The chess master asked that the king should put 1 grain of wheat on the first square of the chess board, twice as much on the second one, twice as much than that on the third one, and so on for all of the 64 squares. Was the king happy with this modest request? Maybe, initially he was, but not so upon reflection of the facts. What do you think?

Section 2. *Sequences given by linear recurrence relations*

Besides progressions there are other well known examples of sequences given by recurrence relations. The most famous is the Fibonacci sequence mentioned above, satisfying the second order linear recurrence relation $a_{n+2} = a_{n+1} + a_n$ and the initial conditions $a_1 = 1$, $a_2 = 1$. Its first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$$

What can you say about the general term? Can we find a formula for a_n? It is not immediately obvious how to answer this question, and this is how we work:
The crucial observation is that the stated recurrence relation and the first two terms *completely determine* the sequence. Indeed, if (b_n) is a sequence satisfying $b_{n+2} = b_{n+1} + b_n$ and $b_1 = a_1$, $b_2 = a_2$ then we claim that $b_n = a_n$ for all n (not just n = 1 or n = 2). Indeed, we have

$$b_3 = b_2 + b_1 = a_2 + a_1 = a_3,$$

hence also $$b_4 = b_3 + b_2 = a_3 + a_2 = a_4,$$

and generally, by a simple inductive argument, $b_n = a_n$ for all n.

To conclude, we shall seek a sequence (b_n) satisfying $b_{n+2} = b_{n+1} + b_n$ and $b_1 = a_1 = 1$, $b_2 = a_2 = 1$.
One way to argue is to find a (non-zero) solution of form $b_n = r^n$ of the stated second order linear recurrence relation.
Substituting $b_n = r^n$ in $b_{n+2} = b_{n+1} + b_n$ and canceling the common factor we find that r must satisfy the so called *characteristic equation*

$$r^2 = r + 1.$$

This equation has two roots,

$$r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}.$$

Thus, both $b_n = r_1^n = \left(\frac{1+\sqrt{5}}{2}\right)^n$ and $b_n = r_2^n = \left(\frac{1-\sqrt{5}}{2}\right)^n$ satisfy the recurrence $b_{n+2} = b_{n+1} + b_n$. This should be clear, but let us give a proof:
For $b_n = r_1^n$ we have

$$b_{n+2} - (b_{n+1} + b_n) = r_1^{n+2} - \left(r_1^{n+1} + r_1^n\right) = r_1^n\left(r_1^2 - r_1 - 1\right) = r_1^n \cdot 0 = 0.$$

Similarly for the case of $b_n = r_2^n$.
Note that, due to the linearity of the recurrence, yet another solution is

$$b_n = Ar_1^n + Br_2^n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n \qquad (*)$$

for any choice of constants A, B. This is easily checked directly and we leave it to the reader. Thus we seek A and B so that the initial conditions $b_1 = a_1 = 1$, $b_2 = a_2 = 1$ are also satisfied.
Putting n = 1 and n = 2 in (*) we find

$$A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

and

$$A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1$$

Solving the system for A and B we will find $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$,
By the remarks above we conclude that

$$a_n = b_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

In other words we found a formula for the general term of the Fibonacci sequence.

Here is another example: Find the general term of the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ ($n \geq 2$) subject to the initial conditions $a_0 = 3$ and $a_1 = 7$.
As before, trying a solution of the form $a_n = r^n$ substituted in the recurrence gives (after cancellation) the characteristic equation

$$r^2 = 3r - 2.$$

This has roots $r_1 = 1$ and $r_2 = 2$. Thus we expect a solution of the form $a_n = Ar_1^n + Br_2^n = A + B\cdot 2^n$, where the constants A, B are determined from the initial conditions $a_0 = 3$ and $a_1 = 7$. This is the subject of the next problem.

Problem 8. Show that $a_n = A + B\cdot 2^n$ (n = 0, 1, 2, ...) is a solution of the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ (n ≥ 2). Determine the values of the constants A and B that make this formula the specific solution of the given recurrence relation with the initial conditions $a_0 = 3$ and $a_1 = 7$.

Solution (direct, not using the theory developed except when necessary). First we plug the proposed solution into the recurrence relation. The left-hand side a_n is just $A + B\cdot 2^n$. To write down the right-hand side we note that the proposed solution gives us $a_{n-1} = A + B\cdot 2^{n-1}$ and $a_{n-2} = A + B\cdot 2^{n-2}$. Thus the right-hand side $3a_{n-1} - 2a_{n-2}$ is $3(A + B\cdot 2^{n-1}) - 2(A + B\cdot 2^{n-2})$. Therefore, we want to verify that the equation

$$A + B\cdot 2^n = 3(A + B\cdot 2^{n-1}) - 2(A + B\cdot 2^{n-2})$$

holds for all $n \geq 2$. This is a simple exercise in algebra. Starting from the right-hand side we have

$$3(A + B\cdot 2^{n-1}) - 2(A + B\cdot 2^{n-2}) = 3A + 3B\cdot 2^{n-1} - 2A - 2B\cdot 2^{n-2} =$$

$$= A + B\cdot 2^{n-2}(3\cdot 2 - 2) = A + 4B\cdot 2^{n-2} = A + B\cdot 2^n$$

which is exactly the left-hand side.
Now we find the appropriate A and B so that the initial conditions $a_0 = 3$, $a_1 = 7$ are also satisfied.

Plugging $n = 0$ into our solution and invoking the given condition that $a_0 = 3$, we obtain the equation $3 = A + B\cdot 2^0 = A + B$. Plugging $n = 1$ into our solution and invoking $a_1 = 7$, we obtain the equation $7 = A + B\cdot 2^1 = A + 2B$. Thus we need to solve the system of simultaneous linear equations

$$\begin{cases} A + B = 3 \\ A + 2B = 7 \end{cases}$$

An easy exercise in elementary algebra yields $A = -1$ and $B = 4$. Therefore, the unique solution of the recurrence relation with the stated initial conditions is $a_n = -1 + 4\cdot 2^n$. This tells us, for instance, that $a_7 = -1 + 4\cdot 2^7 = 511$, without having to compute the terms of the sequence one by one until we reach a_7.

Let us summarize: From the two examples above it should be clear that to solve a second order linear recurrence of the from $a_{n+2} + pa_{n+1} + qa_n = 0$ with a_1, a_2 given we first solve the quadratic equation $r^2 + pr + qr = 0$. If it has *two distinct roots* r_1 and r_2 , then the general term is of the form $a_n = Ar_1^n + Br_2^n$. The constants A and B are determined from a_1, a_2 by solving a system. In fact this idea can be generalized to linear recurrence relations

$$a_{n+k} + p_1 a_{n+k-1} + ... + p_k a_n = 0$$

of k degree, with k initial conditions given, for a long as the k^{th} degree characteristic equation

$$r^k + p_1 r^{k-1} + ... + p_k = 0$$

that arises, has k distinct roots. Then the general term is of the form

$$a_n = A_1 r_1^n + A_2 r_2^n + ... + A_k r_k^n$$

Section 3. *Monotonic sequences, bounded sequences*

In this and the next section we shall study some properties sequences may or may not have, such as monotonicity, boundedness or convergence. The exact definitions will be given below. In the end we shall be able to answer questions such as

Question. Consider now the sequence:

$$\sqrt{2}\,,\ \sqrt{2+\sqrt{2}}\,,\ \sqrt{2+\sqrt{2+\sqrt{2}}}\,,\ ...\,,\ \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n\,\text{radicals}}\,,\ ...$$

Is this an increasing sequence? Is this a bounded sequence? Does the sequence a limit?

Here are some definitions.

Definition. A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be *non-decreasing* if $a_n \le a_{n+1}$ for all indices n. If the inequality is strict, then it is called *increasing*. Similarly we define *non-increasing* sequences (if $a_n \ge a_{n+1}$ for all indices n), and *decreasing* ones. Sequences which are either non-decreasing or non-increasing, are called *monotonic*.

Definition. A sequence $(a_n)_{n\in\mathbf{N}}$ is said to be *bounded above* if there is a number M such that $a_n \le M$ for all indices n. Such a number M is called an *upper bound* of the sequence. Similarly, the sequence is said to be *bounded below* if there is a number m such that $a_n \ge m$ for all indices n. Such a number m is called an *lower bound* of the sequence. Finally, a sequences is said to be *bounded* if it is at the same time both upper bounded and lower bounded.

For example the sequence $a_n = n$ is a) increasing, b) bounded below and c) *not* bounded above. Similarly, the constant sequence $b_n = 1$ is a) non-decreasing, b) non-increasing and c) bounded.
Note that an upper bound of a sequence, if there is one is not unique: If M is an upper bound, so are $M + \frac{1}{2}$ or M + 1, to name just a few. Similarly for lower bounds.

Problem 9. Show that a sequence (a_n) is bounded if and only if there is an M > 0 such that

$$|a_n| < M \text{ (for all n)}$$

Problem 10. Study the monotonicity of the following sequences:

a) $a_n = \dfrac{1}{n}$, b) $b_n = 1 - \dfrac{1}{n}$, c) $c_n = \dfrac{n}{n+1}$, d) $d_n = 3\cdot\left(\dfrac{2}{3}\right)^n$,

e) $e_n = (-1)^n \frac{1}{n}$, f) $f_n = 1 + (-1)^n \frac{1}{n}$, g) $g_n = (-1)^n \frac{n}{n+1}$, h) $h_n = 3 \cdot \left(-\frac{2}{3}\right)^n$.

Problem 11. Study the monotonity of the following sequences:

a) $a_n = \frac{n^2}{n+1}$, b) $b_n = n - \frac{1}{n}$, c) $c_n = \frac{n^2}{n^2+1}$, d) $d_n = 3 \cdot \left(\frac{5}{3}\right)^n$

e) $e_n = (-1)^n \sqrt{n^2+1}$, f) $f_n = n + (-1)^n \frac{1}{n}$, g) $g_n = 2^n (-1)^n \frac{n}{n+1}$,

h) $h_n = \frac{n^2}{n^2+1}\left(\frac{2}{3}\right)^n$.

Problem 12. Study whether the following sequences are bounded, or not:

a) $a_n = \frac{1}{n}$, b) $a_n = 1 - \frac{1}{n}$, c) $a_n = \frac{n}{n+1}$, d) $a_n = 3 \cdot \left(\frac{2}{3}\right)^n$,

e) $a_n = (-1)^n \frac{1}{n}$, f) $a_n = 1 + (-1)^n \frac{1}{n}$, g) $a_n = (-1)^n \frac{n}{n+1}$, h) $a_n = 3 \cdot \left(-\frac{2}{3}\right)^n$.

Problem 13. Study whether the following sequences are bounded, or not:

a) $a_n = \frac{n^2}{n+1}$, b) $b_n = n - \frac{1}{n}$, c) $c_n = \frac{n^2}{n^2+1}$, d) $d_n = 3 \cdot \left(\frac{5}{3}\right)^n$

e) $e_n = (-1)^n \sqrt{n^2+1}$, f) $f_n = n + (-1)^n \frac{1}{n}$, g) $g_n = 2^n (-1)^n \frac{n}{n+1}$,

h) $h_n = \frac{n^2}{n^2+1}\left(\frac{2}{3}\right)^n$.

Problem 14. Show that the sequence (s_n), where s_n = $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$, is not bounded above. You may use the inequality x ≥ log (1 + x) for x ≥ 0 (which is easily proved using calculus).

Solution. From the given inequality we have

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} \geq \log(1 + 1) + \log\left(1 + \frac{1}{2}\right) + \log\left(1 + \frac{1}{3}\right) + \ldots + \log\left(1 + \frac{1}{n}\right)$$

$$= \log\left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{n+1}{n}\right)$$

$$= \log (n + 1)$$

so it is not bounded above.

Sequences given by recurrence relations can also be tested for monotonicity or boundednes. Here is an example.

Problem 15. Prove that the sequence given by $a_{n+1} = \sqrt{2 + a_n}$, where $a_1 = \sqrt{2}$, is a bounded, increasing sequence.

Solution. For positive sequences it is enough to find an upper bound (0 is a lower bound).

We observe that $a_1 = \sqrt{2} < 2$, hence $a_2 = \sqrt{2 + \sqrt{2}} < \sqrt{2+2} = 2$.

We proceed by induction, and suppose that $a_n < 2$, for n = k . Then for n = k + 1 we have

$$a_{k+1} = \sqrt{2 + a_k} < \sqrt{2+2} = 2,$$

hence, by induction, the sequence is bounded.

To prove monotonicity we have to compare $a_{n+1} = \sqrt{2 + a_n}$ with a_n. Since $a_2 > a_1$ we try to see whether $a_{n+1} > a_n$ for all n.

Squaring both sides we see that $\sqrt{2 + a_n} > a_n$ is equivalent to $2 + a_n > a_n^2$, or $0 > a_n^2 - a_n - 2$. Now, the parabola $x^2 - x - 1$ (this stands on the right hand side) has roots $x_1 = -1$, $x_2 = 2$, hence for all $-1 < x < 2$ it will be negative. But we saw before that $0 < a_n < 2$, for all natural numbers n, and so $0 > a_n^2 - a_n - 2$. Hence $a_{n+1} > a_n$ as required.

(Note that once we proved that the sequence is increasing, the bounds $0 < a_n < 2$ can be improved to $\sqrt{2} < a_n < 2$, for all natural numbers.

We remark that the sequence studied in the previous problem is precisely the sequence appearing in the *Question* at the beginning of this Section. Thus we answered two of the questions stated there. It remains to discuss whether it has a limit. This is the topic studied in the next Section.

Section 4. Convergence of sequences

Next we discuss the convergence of sequences. Roughly speaking (the exact definition follows) a sequence (a_n) "approaches" or "converges" to a limit L, as n increases, if given any "tolerance" $\varepsilon > 0$, the terms of the sequence eventually differ from L by less than ε. More precisely,

Definition. We say that the sequence (a_n) *converges* to a real number L, written as $\lim_{n\to\infty} a_n = L$, if for any given $\varepsilon > 0$, there exists a positive integer N such that for all $n \geq N$, we have

$$|a_n - L| < \varepsilon \quad . \tag{1}$$

If a sequence is not convergent, it is called *divergent*. The number L is said to be the *limit* of the sequence (a_n).

Instead of the notation $\lim_{n\to\infty} a_n = L$ we sometimes use the shorter notation $\lim a_n = L$ or the alternative $a_n \xrightarrow{n} L$.

Some times is more convenient to view the inequalities in (1) in the equivalent form

$$L - \varepsilon < a_n < L + \varepsilon$$

Note that if (a_n) converges, the N stated in the definition is not unique: If a particular N is sufficient to show (1) for all $n \geq N$, then any larger one is also sufficient. Note further that, generally, N depends on ε. Changing ε may result in a different choice for N.

Another thing to observe is that it follows immediately from the definition that if the sequence (a_n) converges then so does the sequence (a_{n+1}) and in fact $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$.

Iterating this we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_{n+k}$ for any fixed $k \in \mathbf{N}$.

Examples. a) The constant sequence (a_n), where $a_n = c$ for all n, converges. In fact $\lim_{n\to\infty} a_n = c$.

b) The sequence $\left(\frac{1}{n}\right)$ converges to 0. In symbols, $\lim_{n\to\infty} \frac{1}{n} = 0$.

Proof. a) This is obvious since, given $\varepsilon > 0$, for *any* N we have for all $n \geq N$ that

$$c - \varepsilon < c < c + \varepsilon$$

b) Let $\varepsilon > 0$ be given. Take for N any integer larger than $\frac{1}{\varepsilon}$. For example N = $\left[\frac{1}{\varepsilon}\right] + 1$ would do. Then for $n \geq N = \left[\frac{1}{\varepsilon}\right] + 1 > \frac{1}{\varepsilon}$ we have $\frac{1}{n} < \varepsilon$. As n > 0 we in fact have, for all $n \geq N$,

$$\left|\frac{1}{n} - 0\right| < \varepsilon.$$

By definition, then, $\lim_{n\to\infty} \frac{1}{n} = 0$.

The next two problems can by solved easily using the theorems we will develop later. However the reader is asked to solve them now using directly the definition.

Problem 16. Prove that the sequences $\frac{2}{n}$, $\frac{1}{n^2}$ and $\frac{1}{\sqrt{n}}$ are convergent to 0.

Problem 17. Prove that $\lim \frac{n}{n+1} = 1$, $\lim \frac{n^2-1}{n^2+1} = 1$ and $\lim \frac{2-3n}{n+1} = -\frac{3}{2}$.

Problem 18. Show that the sequence $a_n = (-1)^n$ does not converge to 0. More generally, show that it does not converge to any number L.

(Hint for the first part. Show that for $\varepsilon = \frac{1}{2} > 0$ it is *not* possible to find N such that for all $n \geq N$ we have $-\varepsilon < (-1)^n < \varepsilon$).

Problem 19. Prove that if a sequence (a_n) satisfies $\lim\limits_{n\to\infty} a_n = L$, then $\lim\limits_{n\to\infty} |a_n| = |L|$. Is the converse true?

Solution. This is immediate using the inequality 0 ≤ $\big||a_n| - |L|\big| \le |a_n - L|$. The converse is false as the example of the non-convergent $a_n = (-1)^n$ shows, in which $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} |(-1)^n| = \lim\limits_{n\to\infty} 1 = 1$.

Before giving more problems concerning convergent sequences, let us prove some useful theorems.

Theorem 1. Every convergent sequence is bounded.

Proof. Let (a_n) be a sequence with $\lim\limits_{n\to\infty} a_n = L$. Apply the definition of convergence for $\varepsilon = 1$. For this ε > 0, there is an N such that for all $n \ge N$ we have $|a_n - L| < 1$, and so $|a_n| < |L| + 1$. Set M = max $\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |L| + 1\}$. It is now clear that for all n we have $|a_n| \le M$.

Note that the converse of this theorem is false, as the sequence $(-1)^n$ is bounded but not convergent.

Problem 20. Suppose that (a_n) is a sequence of non-zero terms with $\lim\limits_{n\to\infty} a_n = L$, where L ≠ 0. Show that the sequence $\left(\dfrac{1}{a_n}\right)$ is bonded. (Using this result we shall show below that actually the later sequence is convergent).

(Hint. Set $\varepsilon = \dfrac{|L|}{2}$ > 0. Then for some N and all n ≥ N we have $|a_n - L| < \varepsilon = \dfrac{|L|}{2}$ and so $|a_n| > \dfrac{|L|}{2}$).

Theorem 2. If (a_n), (b_n) and (c_n) are three sequences such that $a_n \le b_n \le c_n$ for all n and such that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$. Then also $\lim\limits_{n\to\infty} b_n = L$.

Proof. Let $\varepsilon > 0$ be given. As $\lim\limits_{n\to\infty} a_n = L$, there exists an N_1 such that for n ≥ N_1 we have

$$L - \varepsilon < a_n \qquad (1)$$

(actually we have $L - \varepsilon < a_n < L + \varepsilon$, but we shall not use the second inequality). Similarly there exists an N_2 such that for n ≥ N_2 we have

$$c_n < L + \varepsilon. \qquad (2)$$

Set N = max (N_1 , N_2). Then for all n ≥ N, both (1) and (2) hold giving

$$L - \varepsilon < a_n \le b_n \le c_n < L + \varepsilon .$$

In other words, for all n ≥ N we have $L - \varepsilon < b_n < L + \varepsilon$ implying that (b_n) converges and that $\lim_{n\to\infty} b_n = L$.

Theorem 3. Suppose that (a_n), (b_n) are sequences with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ and suppose p, q are constants.

Then $\lim_{n\to\infty}(pa_n + qb_n) = pA + qB$ and $\lim_{n\to\infty}(a_n b_n) = AB$. If also $b_n \ne 0$ for all n and $B \ne 0$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$.

Proof. Let $\varepsilon > 0$ be given. Note that also $\frac{\varepsilon}{2|p|+1}$ is > 0. Using this in the definition of $\lim_{n\to\infty} a_n = L$, there exists an N_1 such that for n ≥ N_1 we have

$$| a_n - A | < \frac{\varepsilon}{2|p|+1} \qquad (1)$$

Similarly there exists an N_2 such that for n ≥ N_2 we have

$$| b_n - B | < \frac{\varepsilon}{2|q|+1} \qquad (2)$$

Set N = max (N_1 , N_2). Then for all n ≥ N, both (1) and (2) hold giving

$$\begin{aligned} | pa_n + qb_n - (pA + qB) | &< | pa_n - pA | + | qb_n - qB | \\ &< \frac{|p|}{2|p|+1}\varepsilon + \frac{|q|}{2|q|+1}\varepsilon \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \\ &= \varepsilon \end{aligned}$$

Hint for a proof of $\lim_{n\to\infty}(a_n b_n) = AB$: Use

$$\begin{aligned} | a_n b_n - AB | = | (a_n - A)b_n + A(b_n - B) | &\le | a_n - A | \cdot | b_n | + | A | \cdot | b_n - B | \\ &\le | a_n - A | \cdot M + | A | \cdot | b_n - B | \end{aligned}$$

where M is an upper bound of the (convergent) sequence (b_n).

Hint for the proof of $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$: First note that, by the previous, it is sufficient to prove $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{B}$. Use now $\left|\frac{1}{b_n} - \frac{1}{B}\right| = \left|\frac{B - b_n}{Bb_n}\right| \le M \frac{| B - b_n |}{B}$ where M is an upper bound of $\left(\frac{1}{b_n}\right)$, which exists according to Problem 20.)

Problem 21. Prove that the sequence $\left(\frac{1}{n}\sin(n^7 - 3n^2 + 17)\right)$ converges to 0.

(Hint. use the fact that $-\frac{1}{n} \le \frac{1}{n}\sin(n^7 - 3n^2 + 17) \le \frac{1}{n}$ and that $\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$.)

Problem 22. Find the limit $\lim_{n\to\infty} \frac{4n^3 + 5n^2 - 11n + 3}{3n^3 - 2n^2 - 9n + 7}$.

Solution. Dividing numerator and denominator by n^3 and making repeated use of Theorem 3, we have

$$\lim_{n\to\infty} \frac{4n^3 + 5n^2 - 11n + 3}{3n^3 - 2n^2 - 9n + 7} = \lim_{n\to\infty} \frac{4 + \frac{5}{n} - \frac{11}{n^2} + \frac{3}{n^3}}{3 - \frac{2}{n} - \frac{9}{n^2} + \frac{7}{n^3}} = \lim_{n\to\infty} \frac{4+0-0+0}{3-0-0+0} = \frac{4}{3}.$$

Problem 23. Let $a_1 = \frac{2}{3}$ and $a_{n+1} = a_n + \frac{1}{(n+1)(n+2)}$ for $n \ge 1$. Find an expression for a_n, and prove that the sequence (a_n) converges.

Solution. Calculating $a_2 = a_1 + \frac{1}{3\cdot 4} = \frac{2}{3} + \frac{1}{3\cdot 4} = \frac{3}{4}$, and $a_3 = a_2 + \frac{1}{4\cdot 5} = \frac{3}{4} + \frac{1}{4\cdot 5} = \frac{4}{5}$, we can "guess" that $a_n = \frac{n+1}{n+2}$. Indeed, this can easily be shown by induction using the recurrence relation, and is left to the reader. It is now easy to show that $\lim_{n\to\infty} \frac{n+1}{n+2} = 1$.

Problem 24. Suppose (a_n) is a sequence of positive terms such that $\lim_{n\to\infty} a_n = L$. Prove that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$ and, more generally, $\lim_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{L}$ for any fixed k $\in$ **N**.

(Hint. For the case L ≠ 0 use $\left|\sqrt{a_n} - \sqrt{L}\right| = \left|\frac{a_n - L}{\sqrt{a_n} + \sqrt{L}}\right| \le \frac{|a_n - L|}{\sqrt{L}}$)

Some very important limits are given in the next three problems.

Problem 25. Prove that for all c with $|c| < 1$ we have $\lim_{n\to\infty} c^n = 0$.

Solution. For c = 0 this is immediate, so we may assume $0 < |c| < 1$. Then for some d > 0 we have $|c| = \frac{1}{1+d}$ so $0 < |c^n| = |c|^n = \frac{1}{(1+d)^n} < \frac{1}{1+nd}$ (this last uses the Bernoulli inequality $(1 + x)^n \ge 1 + nx$, if $x \ge 0$. It can be proved by expanding the binomial on the left. It can also be proved by induction). But as $\lim_{n\to\infty} \frac{1}{1+nd} = 0$, the required conclusion follows from Theorem 3.

Problem 26. Prove that for all a > 0 we have $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

Solution. If a = 1 this is clear. Take a > 1. Then $\sqrt[n]{a} > 1$ so we may write $\sqrt[n]{a} = 1 + d_n$ where $d_n > 0$. Thus $a = (1 + d_n)^n \ge 1 + n\, d_n$ (by the Bernoulli inequality $(1 + x)^n \ge 1 + nx$, for $x \ge 0$). Hence we have

$$0 < d_n < \frac{a-1}{n}$$

As $\lim_{n\to\infty} \frac{a-1}{n} = 0$, Theorem 2 shows that $\lim_{n\to\infty} d_n = 0$. But then, using Theorem 3,

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} (1 + d_n) = 1.$$

For the case 0 < a < 1, consider $\frac{1}{a}$, which is > 1. By what was just proved we have $\lim_{n\to\infty} \frac{1}{\sqrt[n]{a}} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{a}} = 1$. So, inverting and using Theorem 3, $\lim_{n\to\infty} \sqrt[n]{a} = 1$, again.

Problem 27. Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(Hint. Use an argument similar to the one in the previous problem only replace the Bernoulli inequality with the stronger one $(1 + x)^n \geq 1 + nx + \frac{n(n-1)}{2} x^2 \geq 1 + \frac{n(n-1)}{2} x^2$, for x > 0.)

Also, it is important to know (but we shall not prove) that the factorial grows faster than the exponential function in the sense that for any c > 0 we have

$$\lim_{n\to\infty} \frac{c^n}{n!} = 0,$$

and the exponential grows faster than power functions:

$$\lim_{n\to\infty} \frac{n^k}{c^n} = 0.$$

Problem 28. Let (x_n) be an arithmetic progression of positive terms. Study the convergence of the sequences $a_n = \frac{x_{n+1}}{x_n}$ and $s_n = \sum_{k=1}^{n} \frac{x_{k+1}}{kx_k}$.

Solution. For an arithmetic progression we have $x_n = a + (n-1)d$, where $a = x_1$. Note that $d \geq 0$, since the terms are positive. Thus

$$\frac{x_{n+1}}{x_n} = \frac{a + nd}{a + (n-1)d} = 1 + \frac{d}{a + (n-1)d},$$

and so $\lim \frac{x_{n+1}}{x_n} = \lim\left(1 + \frac{d}{a + (n-1)d}\right) = 1$.

For $s_n = \sum_{k=1}^{n} \frac{x_{k+1}}{kx_k}$ we have

$$\sum_{k=1}^{n} \frac{x_{k+1}}{kx_k} = \sum_{k=1}^{n} \frac{1}{k}\left(1 + \frac{d}{a + (n-1)d}\right) \geq \sum_{k=1}^{n} \frac{1}{k}.$$

But the right hand side is not bounded above (see Problem 14). So, by Theorem 1, the sequence (s_n) is divergent.

Problem 29. Let (x_n) be a geometric progression consisting of positive terms. Study the convergence of the sequences $a_n = \dfrac{\log x_{n+1}}{\log x_n}$ and $p_n = \sqrt[n^2]{x_1 x_2 x_3 \ldots x_n} = \sqrt[n^2]{\prod_{k=1}^{n} x_k}$.

Solution. For the terms of a geometric progression we have $x_n = x_1 r^{n-1}$. Here x_1 and r are > 0, We have

$$\lim_{n\to\infty}\frac{\log x_{n+1}}{\log x_n} = \lim_{n\to\infty}\frac{\log x_1 + n\log r}{\log x_1 + (n-1)\log r} =$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n}\log x_1 + \log r}{\frac{1}{n}\log x_1 + \frac{n-1}{n}\log r}$$

$$= \lim_{n\to\infty}\frac{0+\log r}{0+\log r} = 1$$

For the second sequence we have

$$\lim_{n\to\infty}\sqrt[n^2]{\prod_{k=1}^{n} x_k} = \lim_{n\to\infty}\sqrt[n^2]{x_1^n r^{\frac{n(n+1)}{2}}} = \lim_{n\to\infty}\left(\sqrt[n]{x_1}\cdot r^{\frac{1}{2}\cdot\frac{n+1}{n}}\right) = 1\cdot r^{\frac{1}{2}} = \sqrt{r}.$$

Section 5. Monotonicity and convergence

One of the most important theorems on convergence of sequences is the following:

Theorem 4. Every bounded above non-decreasing sequence is convergent.

The proof is beyond the scope of these notes, but it does not mean that it is difficult. What happens is that the proof depends on the axioms that define real numbers, and in particular on the so called *Completeness Axiom*[10].

Using Theorem 4, or directly, it can be shown that every bounded below non-increasing sequence also converges.

Here are some applications of Theorem 4.

For a start we prove that the sequence referred to on the *Question* of Section 3 and studied in Problem 15, is convergent.

Problem 30. Prove that the sequence given by $a_{n+1} = \sqrt{2+a_n}$, where $a_1 = \sqrt{2}$, is convergent. What is its limit?

Solution. We have already shown that the sequence (a_n) is bounded and increasing. By Theorem 4, it converges and it is required to find the limit.

Set $\lim_{n\to\infty} a_n = L$. Now, from the recurrence relation $a_{n+1} = \sqrt{2+a_n}$ and taking limits we have

[10] It states that bounded every non-empty set of real numbers has a *least upper bound* (also called *supremum*). This means that there is an upper bound of the set which is less than or equal any other upper bound. It can be shown that the limit referred to in Theorem 4, of a bounded non-decreasing sequence, is its least upper bound considered as a non-empty bounded set.

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n} \,. \qquad (1)$$

But $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$ (see Section 4, after the definition of convergence). Also, using Theorem 3 and Problem 24 we have

$$\lim_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+L} \,.$$

Comparing the two sides of (1) we conclude that $L = \sqrt{2+L}$. Squaring both sides we get $L^2 - L - 2 = 0$, with the solutions $L_1 = -1$, $L_2 = 2$. The conditions of the problem impose $\sqrt{2} < L \le 2$, hence $L = 2$ is the only possible solution. Thus we conclude $\lim_{n\to\infty} a_n = 2$.

Problem 31. Consider the sequence $a_{n+1} = \sin a_n$, where $a_1 = 1$. Prove that the sequence is decreasing and bounded, hence convergent. What is its limit?

(Hint. Recall the inequality $\sin x < x$, for all $x > 0$. The limit L must satisfy $L \ge 0$ and $L = \sin L$.)

Problem 32. Consider the sequence $a_n = \left(1+\frac{1}{n}\right)^n$. Prove that it is bounded, increasing, and hence convergent.

Proof. By the A.M. – G.M. inequality applied to the n + 1 numbers

$$1,\ \underbrace{1+\frac{1}{n}, 1+\frac{1}{n}, \ldots, 1+\frac{1}{n}}_{n \text{ numbers}}$$

we get $\dfrac{1+n\left(1+\frac{1}{n}\right)}{n+1} > \sqrt[n+1]{\left(1+\frac{1}{n}\right)^n}$ and so $1+\dfrac{1}{n+1} > \sqrt[n+1]{\left(1+\frac{1}{n}\right)^n}$, which gives $a_{n+1} > a_n$, showing that (a_n) is increasing.

To show that (a_n) is bounded, use the binomial expansion to get

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\cdot\left(\frac{1}{n}\right)^3 + \ldots + \frac{n(n-1)(n-2)\ldots 1}{n!}\cdot\left(\frac{1}{n}\right)^n \\
&= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right) \\
&= 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!} \\
&\le 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}} \\
&= 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} < 1 + \frac{1}{1-\frac{1}{2}}
\end{aligned}$$

= 3.
This completes the proof.

The limit of the sequence studied in the previous problem plays a very important role for Analysis. It is often taken as the definition of the famous number e of Euler, which is the base of the Neperian logarithms. Thus, by definition, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.